

Black Hole Radiation with Modified Dispersion Relation in Tunneling Paradigm: Static Frame

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Abstract

Due to the exponential high gravitational red shift near the event horizon of a black hole, it might appear that the Hawking radiation would be highly sensitive to some unknown high energy physics. To study possible deviations from the Hawking's prediction, the dispersive field theory models have been proposed, following the Unruh's hydrodynamic analogue of a black hole radiation. In the dispersive field theory models, the dispersion relations of matter fields are modified at high energies, which leads to modifications of equations of motion. In this paper, we use the Hamilton-Jacobi method to investigate the dispersive field theory models. The preferred frame is the static frame of the black hole. The dispersion relation adopted agrees with the relativistic one at low energies but is modified near the Planck mass m_p . We calculate the corrections to the Hawking temperature for massive and charged particles to $\mathcal{O}(m_p^{-2})$ and massless and neutral particles to all orders. Our results suggest that the thermal spectrum of radiations near horizon is robust, e.g. corrections to the Hawking temperature are suppressed by m_p . After the spectrum of radiations near the horizon is obtained, we use the brick wall model to compute the thermal entropy of a massless scalar field near the horizon of a 4D spherically symmetric black hole. We find that the leading term of the entropy depends on how the dispersion relations of matter fields are modified, while the subleading logarithmic term does not. Finally, the luminosities of black holes are computed by using the geometric optics approximation.

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I. INTRODUCTION

The classical theory of black holes predicts that anything, including light, couldn't escape from the black holes. However, Stephen Hawking demonstrated that quantum effects could allow black holes to radiate a thermal flux of quantum particles[1]. The assumption that the ultra-high energy modes are in their ground state was used to derive the Hawking radiation in the framework of quantum field theory in curved spacetime. After this discovery, it was realized that there was the trans-Planckian problem with the calculation[2]. Due to the exponential high gravitational red shift near the horizon, the outgoing particles of the Hawking radiation originate from the extremely high (e.g., trans-Planckian) frequency modes. So the Hawking radiation relies on the validity of quantum field theory in curved spacetime to arbitrary high energies. On the other hand, quantum field theory is considered more like an effective field theory of an underlying theory whose nature remains unknown[3]. This observation poses the question of whether any unknown physics at the Planck scale could strongly influence the Hawking radiation. It is believed that the trans-Planckian physics manifests itself in certain modifications of the existing models. Thus, even though a complete theory of quantum gravity is not yet available, we can use a “bottom-to-top approach” to probe the possible effects of quantum gravity on our current theories and experiments[4]. One possible way of how such an approach works is via modifications of energy-momentum dispersion relation

$$p^2 = E^2 - m^2, \tag{1}$$

which has been reviewed in the framework of Lorentz violating theories in [5, 6].

To study the trans-Planckian problem, the dispersive field theory models[7–17] have been proposed, which focused on studying the effect on the Hawking radiation due to modifications of the dispersion relations of matter fields at high energies. These models were motivated by a hydrodynamic analogue of a black hole radiation[7]. Similar to the original method for deriving the Hawking radiation, the energy fluxes for outgoing radiation were usually obtained by calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation. In most works, the Hawking effect could be recovered at leading order for some ranges of the black hole's temperature and the particle's frequency under some suitable assumptions although the mechanisms for recovering it were different than in the non-dispersive case. These assumptions and ranges in the models have been

briefly reviewed in [18, 19]. While most works used the free-fall preferred frame to modify the dispersion relation, accelerated frames has been studied in [11]. It has been found that the fluxes emitted by a black hole were not significantly affected by the choice of preferred frame as long as the acceleration of the frame was not too drastic. However, it has been numerically shown that the created particle flux dropped significantly if the acceleration became large. Thus, it was conjectured that there might be no Hawking radiation for a static frame.

There are various methods for deriving the Hawking radiation and calculating its temperature. Among them is a semiclassical method of modeling Hawking radiation as a tunneling process. This method was first proposed by Kraus and Wilczek[20, 21], which is known as the null geodesic method. They employed the dynamical geometry approach to calculate the imaginary part of the action for the tunneling process of s-wave emission across the horizon and related it to the Boltzmann factor for the emission at the Hawking temperature. Later, the tunneling behaviors of particles were investigated using the Hamilton-Jacobi method[22–24]. In the Hamilton-Jacobi method, one ignores the self-gravitation of emitted particles and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. The tunneling probability for the classically forbidden trajectory from inside to outside the horizon is obtained by using the Hamilton-Jacobi equation to calculate the imaginary part of the action for the tunneling process. Using the null geodesic method and Hamilton-Jacobi method, much fruit has been achieved[25–36]. Furthermore, the effects of quantum gravity on the Hawking radiation have been discussed in the Hamilton-Jacobi method. In fact, the minimal length deformed Hamilton-Jacobi equation for fermions in curved spacetime have been introduced and the modified Hawking temperatures have been derived[37–42]. These motivate us to use the Hamilton-Jacobi method to study the dispersive field theory models. Note that a tunneling model has been introduced in dispersive models of analogue gravity in [17]. The Hamilton-Jacobi equations were imported to curved spacetime using the static preferred frame in [37–42], which leads us first to considering the static preferred frame for the dispersive field theory models. The models with free-fall preferred frame will be investigated in [43]. Comparisons between the results in our paper and those in [37–41] will be given at the end of the section II.

The remainder of our paper is organized as follows. In section II, the deformed Hamilton-Jacobi equations are derived for the dispersive models. In section IV, we solve the deformed

Hamilton-Jacobi equations to obtain tunneling rates for massive and charged particles to $\mathcal{O}(m_p^{-2})$ and massless and neutral particles to all orders. Thermodynamics of radiations near the horizon is discussed in section IV. The thermal entropy of a massless scalar field near the horizon is computed in section V using the brick wall model. In section VI, we calculate luminosities of a 4D spherically symmetric black hole with the mass $M \gg m_p$ and a 2D one. Section VII is devoted to our discussion and conclusion, where the limitations of our calculations are discussed. Effective field theories incorporating the MDR are constructed in the appendix to obtain the deformed Hamilton-Jacobi equations. Throughout the paper we take Geometrized units $c = G = 1$, where the Planck constant \hbar is square of the Planck Mass m_p .

II. DEFORMED HAMILTON-JACOBI EQUATION

In most cases, the modified dispersion relation (MDR) could take the form of

$$p^2 = m_p^2 H\left(\frac{E}{m_p}, \frac{m}{m_p}\right), \quad (2)$$

where m_p is Planck mass and $H(x, y) = x^2 - y^2$ for the unmodified dispersion relation. Taylor expanding the right-hand side of eqn. (2) for $E, m \ll m_p$ gives

$$p^2 = \sum_{i,j=0}^{\infty} h_{i,j} \frac{E^i m^j}{m_p^{i+j-2}}, \quad (3)$$

where $h_{i,j}$ is the coefficient of $x^i y^j$ in the Taylor series of $H(x, y)$ evaluated at $(0, 0)$. Since eqn. (3) has to become eqn. (1) when $m_p \rightarrow \infty$, we find

$$h_{0,0} = h_{0,1} = h_{1,0} = h_{1,1} = 0 \text{ and } h_{2,0} = h_{0,2} = 1. \quad (4)$$

After some manipulations, eqn. (3) can be put in the form of

$$p^2 = \alpha\left(\frac{m}{m_p}\right) E^2 - \beta\left(\frac{m}{m_p}\right) m^2 + \gamma\left(\frac{m}{m_p}\right) mE + \sum_{n \geq 3} \frac{C_n \left(\frac{m}{m_p}\right) E^n}{m_p^{n-2}}, \quad (5)$$

where $\alpha(0) = \beta(0) = 1$ and $\gamma(0) = 0$. If the modifications to the dispersion relation are suppressed by some the scale of Lorentz violation Λ , the naturalness in effective field theories would imply that $C_n \sim \left(\frac{m_p}{\Lambda}\right)^n$. For $\Lambda \ll m_p$, C_n could become much large. To include a

broader class, we consider a static black hole in the possible presence of electromagnetic potential A_μ with the line element

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - C(r^2) h_{ab}(x) dx^a dx^b, \quad (6)$$

where $f(r)$ has a simple zero at $r = r_h$ with $f'(r_h)$ being finite and nonzero. The vanishing of $f(r)$ at point $r = r_h$ indicates the presence of an event horizon. We also assume that the vector potential A_μ is given by

$$A_\mu = A_t(r) \delta_{\mu t}, \quad (7)$$

which is true for charged static black holes in most cases.

The MDR breaks the Lorentz invariance in flat spacetime. Thus, one needs to pick up a preferred frame to determine the form of the MDR. The energy and momentum in eqn. (5) are defined with respect to the preferred frame, where can be described by the unit vector u^μ tangent to the observers' world lines. Explicitly, we have

$$\begin{aligned} E &= p_\mu u^\mu, \\ p^2 &= E^2 - p_\mu p^\mu, \end{aligned} \quad (8)$$

where p_μ is the energy-momentum vector and E and p are the energy and the norm of the momentum measured in the preferred reference frame, respectively. When introducing the MDR into curved spacetime, we use the vector field $u^\mu(x_\nu)$. To obtain the deformed Hamilton-Jacobi equation incorporating the MDR, it is necessary to specify the profile of the preferred frame in the black hole spacetime. One of natural frames is a static frame hovering above the black hole. For such a frame, the vector field $u^\mu(x_\nu)$ is

$$u^\mu(x_\nu) = \left(\sqrt{g^{tt}}, \vec{0} \right) = \left(\frac{1}{\sqrt{f(r)}}, \vec{0} \right). \quad (9)$$

Plugging eqn. (9) into eqns. (8), one finds that the energy and the magnitude of the momentum becomes

$$\begin{aligned} E &= \frac{p_t}{\sqrt{f(r)}}, \\ p^2 &= f(r) p_r^2 + \frac{h^{ab}(x)}{C(r^2)} p_a p_b. \end{aligned} \quad (10)$$

It can be shown that, if the classical action I is a solution of the Hamilton-Jacobi equation, then the transformation equations give

$$p_\mu = -\partial_\mu I, \quad (11)$$

where $-$ appears since $p_\mu = (E, -\vec{p})$ in our metric signature. Furthermore, since ∂_t is a Killing vector of the background spacetime, $(\partial_t)^\mu p_\mu = p_t$ is a constant. In fact, p_t is the conserved energy of the particle and we define $\omega \equiv p_t = -\partial_t I$, which means we can separate t from other variables when solving for I . Relating I to p_μ via eqn. (11) and putting eqns. (10) into eqn. (5) give the deformed Hamilton-Jacobi equation. In the appendix, the deformed Hamilton-Jacobi equation is also derived in a more rigorous way, specifically in the language of the effective field theory. We show there that if a scalar/fermion obeys the MDR given in eqn. (5) in flat spacetime, the deformed scalar/fermionic Hamilton-Jacobi equation with respect to the preferred static frame in the black hole background spacetime can be both written as

$$X^2 = \alpha \left(\frac{m}{m_p} \right) T^2 - \beta \left(\frac{m}{m_p} \right) m^2 + \gamma \left(\frac{m}{m_p} \right) mT + \sum_{n \geq 3} \frac{C_n \left(\frac{m}{m_p} \right) T^n}{m_p^{n-2}}, \quad (12)$$

where we define

$$T = -\frac{\partial_t I + qA_t}{\sqrt{f(r)}}, \quad X^2 = f(r) (\partial_r I)^2 + \frac{h^{ab}(x) \partial_a I \partial_b I}{C(r^2)}, \quad (13)$$

A_μ is the black hole's electromagnetic potential and q is the particle's charge.

III. TUNNELING RATE

In this section, we use the Hamilton-Jacobi method to investigate the particles' tunneling across the event horizon $r = r_h$ of the metric (6) by solving eqn. (12). Taking into account $\partial_t I = -\omega$, we can employ the following ansatz for the action I

$$I = -\omega t + W(r) + \Theta(x), \quad (14)$$

where ω is the particle's energy. Plugging the ansatz into eqn. (13), we have

$$T = \frac{\omega - qA_t(r)}{\sqrt{f(r)}},$$

$$X^2 = f(r) [\partial_r W(r)]^2 + \frac{h^{ab}(x) \partial_a \Theta(x) \partial_b \Theta(x)}{C(r^2)}. \quad (15)$$

The method of separation of variables gives the differential equation for $\Theta(x)$

$$h^{ab}(x) \partial_a \Theta(x) \partial_b \Theta(x) = \lambda, \quad (16)$$

where λ is a constant and determined by $h^{ab}(x)$. Thus, one has

$$X^2 = f(r) [\partial_r W(r)]^2 + \frac{\lambda}{C(r^2)}, \quad (17)$$

and eqn. (12) becomes an ordinary differential equation for $W(r)$. In this section, we solve eqn. (12) for $\partial_r W(r)$, calculate its residue at $r = r_h$ and find the imaginary part of I which gives the tunneling rate Γ across the event horizon. We will calculate $\text{Im } W$ for two cases, a massive and charged particle to $\mathcal{O}(m_p^{-2})$ and a neutral and massless particle to all orders.

A. Massive and Charged Particle to $\mathcal{O}(m_p^{-2})$

Consider a particle with the mass m and the charge q . Solving eqn. (12) for $p_r \equiv \partial_r W(r)$ gives

$$p_r^\pm = \frac{\pm 1}{\sqrt{f(r)}} \left(\alpha \left(\frac{m}{m_p} \right) \frac{\tilde{\omega}^2(r)}{f(r)} - \beta \left(\frac{m}{m_p} \right) m^2 - \frac{\lambda}{C(r^2)} + m\gamma \left(\frac{m}{m_p} \right) \frac{\tilde{\omega}(r)}{\sqrt{f(r)}} \right)^{\frac{1}{2}} \\ \left(1 + \frac{1}{\alpha \left(\frac{m}{m_p} \right) \frac{\tilde{\omega}^2(r)}{f(r)} - \beta \left(\frac{m}{m_p} \right) m^2 - \frac{\lambda}{C(r^2)} + m\gamma \left(\frac{m}{m_p} \right) \frac{\tilde{\omega}(r)}{\sqrt{f(r)}}} \sum_{n \geq 3} \frac{C_n \left(\frac{m}{m_p} \right) \tilde{\omega}^n(r)}{m_p^{n-2} f(r)^{\frac{n}{2}}} \right)^{\frac{1}{2}}, \quad (18)$$

where $+/-$ denotes the outgoing/ingoing solutions and $\tilde{\omega}(r) \equiv \omega - qA_t(r)$. Here, we have a pole at $r = r_h$. Using the residue theory for the semi circle^[1], we get

$$\text{Im } W_\pm(r) = \pm \frac{\sqrt{\alpha} \tilde{\omega}(r_h) \pi}{2\kappa} (1 + \Delta_{qm} + \mathcal{O}(m_p^{-3})), \quad (19)$$

where we define $\kappa = f'(r_h)/2$ and

$$\Delta_{qm} = -\frac{C_3 m \gamma}{4m_p \alpha^2} + \frac{1}{32m_p^2 \kappa \alpha^4} \left[(24C_4 \alpha - 6C_3^2) \tilde{\omega}'(r_h) \tilde{\omega}(r_h) \alpha^2 + \kappa m^2 \gamma^2 \frac{12C_4 \alpha - 15C_3^2}{2} \right. \\ \left. - \kappa \alpha (6C_3^2 - 8C_4 \alpha) \left(\frac{\lambda}{C(r_h^2)} + \beta m^2 \right) + \alpha^2 (C_3^2 - 4C_4 \alpha) \frac{f''(r_h) \tilde{\omega}^2(r_h)}{\kappa} \right]. \quad (20)$$

The argument $\frac{m}{m_p}$ is suppressed for $\alpha \left(\frac{m}{m_p} \right)$, $\beta \left(\frac{m}{m_p} \right)$, $\gamma \left(\frac{m}{m_p} \right)$ and $C_n \left(\frac{m}{m_p} \right)$ in eqns. (19) and (20).

[1] This procedure will be discussed in detail later in this section.

B. Massless and Neutral Particle to All Orders

We now work with a particle with $m = 0$ and $q = 0$. Solving eqn. (12) for p_r gives

$$p_r^\pm = \frac{\pm 1}{\sqrt{f(r)}} \left(\frac{\omega^2}{f(r)} - \frac{\lambda}{C(r^2)} \right)^{\frac{1}{2}} \left(1 + \frac{1}{\frac{\omega^2}{f(r)} - \frac{\lambda}{C(r^2)}} \sum_{n \geq 3} \frac{C_n}{m_p^{n-2}} \frac{\omega^n}{f(r)^{\frac{n}{2}}} \right)^{\frac{1}{2}}, \quad (21)$$

where $C_n \equiv C_n(0)$ and we use $\alpha(0) = 1$. To get the residue of p_r^\pm at $r = r_h$, we first define a few coefficients C_n^α , $\tilde{C}_{m,n}$, and η_l^k as follows

$$\begin{aligned} (1+x)^\alpha &= \sum_{n \geq 0} C_n^\alpha x^n, \\ \left(\sum_{n=0}^{\infty} C_{n+3} x \right)^m &= \sum_{n=0}^{\infty} \tilde{C}_{m,n} x^n, \\ \eta_l^k &= \sum_{m=0}^k (-1)^l C_m^{\frac{1}{2}} C_l^{-m+\frac{1}{2}} \tilde{C}_{m,k-m}, \end{aligned} \quad (22)$$

where m is a non-negative integer and C_n^α are generalized binomial coefficients with $C_n^\alpha = \prod_{k=1}^n \frac{\alpha-k+1}{k}$. Therefore, one has

$$\begin{aligned} p_r^\pm &= \pm \sum_{m=0}^{\infty} \frac{\omega^m}{m_p^m f^{\frac{m}{2}}(r)} \frac{C_m^{\frac{1}{2}} \omega}{f(r)} \left(1 - \frac{f(r) \lambda}{C(r^2) \omega^2} \right)^{-m+\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{C_{n+3}}{m_p^n} \frac{\omega^n}{f^{\frac{n}{2}}(r)} \right)^m \\ &= \pm \frac{\omega}{f(r)} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^l}{C^l(r^2)} \frac{\eta_l^k}{m_p^k} \frac{\omega^{k-2l}}{f^{\frac{k}{2}-l}(r)} \\ &\sim \pm \frac{\omega}{f(r)} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^l}{C^l(r^2)} \frac{\eta_l^{2k+2l}}{m_p^{2k+2l}} \frac{\omega^{2k}}{f^k(r)} \\ &\sim \pm \omega \sum_{l=0}^{\infty} \left[\frac{\lambda}{C(r_h^2) m_p^2} \right]^l \sum_{k=0}^{\infty} \eta_l^{2k+2l} \frac{\omega^{2k}}{m_p^{2k}} \left(\frac{C^l(r_h^2)}{C^l(r^2)} \frac{1}{f^{k+1}(r)} \right), \end{aligned} \quad (23)$$

where we only keep terms contributing to the residue and set $k \rightarrow k + 2l$ in the third line.

Furthermore, we denote the residue of $\frac{C^l(r_h^2)}{C^l(r^2)} \frac{1}{f^{k+1}(r)}$ at $r = r_h$ by

$$\text{Res} \left(\frac{C^l(r_h^2)}{C^l(r^2)} \frac{1}{f^{k+1}(r)}, r_h \right) = \frac{\zeta_k^l}{2\kappa}. \quad (24)$$

Using the residue theory for the semi circle, one has

$$\text{Im } W_\pm(r) = \pm \frac{\omega \pi}{2\kappa} (1 + \Delta), \quad (25)$$

where

$$\Delta = \sum_{l+k \geq 1} \left[\frac{\lambda}{C(r_h^2) m_p^2} \right]^l \eta_l^{2k+2l} \zeta_k^l \frac{\omega^{2k}}{m_p^{2k}}. \quad (26)$$

Note that $\eta_0^0 = \zeta_0^0 = 1$.

C. Calculating λ

It is easy to see that λ depends on $h_{ab}(x)$. Here we consider two kinds of black holes, 4D cylindrically and spherically symmetric black holes. For a 4D cylindrically symmetric black hole, we have

$$h_{ab}(x) dx^a dx^b = d\theta^2 + \alpha^2 dz^2, \quad (27)$$

where $-\infty < z < \infty$, $0 \leq \theta \leq 2\pi$ and α is some constant. Since ∂_θ and ∂_z are the Killing fields of the background spacetime, we can separate the variables and consider a solution for eqn. (16) of the form

$$\Theta = J_\theta \theta + J_z z, \quad (28)$$

where J_θ and J_z are constant and J_θ is the angular momentum along z -axis. The periodicity of θ gives $J_\theta = n\hbar$ with $n \in \mathbb{Z}$. Thus, one finds

$$\lambda = J_\theta^2 + \frac{J_z^2}{\alpha^2}, \quad (29)$$

where $J_\theta = n\hbar$ with $n \in \mathbb{Z}$.

For a 4D spherically symmetric black hole, we have

$$h_{ab}(x) dx^a dx^b = d\theta^2 + \sin^2 \theta d\phi^2, \quad (30)$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. ∂_ϕ is the Killing vector so we consider a solution for eqn. (16) of the form

$$\Theta = Y(\theta) + J_\phi \phi, \quad (31)$$

where J_ϕ is the angular momentum along z -axis. The periodicity of ϕ gives $J_\phi = m\hbar$ with $m \in \mathbb{Z}$. Since the magnitude of the angular momentum of the particle L can be expressed in terms of $p_\theta \equiv \partial_\theta Y(\theta)$ and J_ϕ ,

$$L^2 = p_\theta^2 + \frac{J_\phi^2}{\sin^2 \theta}, \quad (32)$$

eqn. (16) gives $\lambda = L^2$. Putting eqn. (31) into eqn. (16), one gets

$$p_\theta = \sqrt{\lambda - \frac{m^2 \hbar^2}{\sin^2 \theta}}. \quad (33)$$

On the other hand, the Sommerfeld quantization for p_θ gives

$$\oint d\theta p_\theta = 2\pi \left(n + \frac{1}{2} \right) \hbar, \quad (34)$$

where n is a non-negative integer. The integral in eqn. (34) is calculated in the classically allowed region where p_θ is real, which requires that $\lambda > m^2 \hbar^2$. Integrating the quantization integral, one finds that eqn. (34) becomes

$$2\pi \left(\sqrt{\lambda} - |m| \hbar \right) = 2\pi \left(n + \frac{1}{2} \right) \hbar. \quad (35)$$

Solving eqn. (35) for λ gives the WKB leading quantization of the angular momentum

$$\lambda = \left(l + \frac{1}{2} \right)^2 \hbar^2, \quad (36)$$

where $l = n + |m| = 0, 1, \dots$ with $|m| \leq l$. Note that the difference between the exact quantization of the angular momentum $L^2 = l(l+1)\hbar^2$ and the WKB leading quantization $L^2 = \left(l + \frac{1}{2} \right)^2 \hbar^2$ is $\frac{\hbar^2}{4}$.

D. Tunneling Rate

When one calculates the quantum tunneling rate from $\text{Im } W_\pm$, there is so called “factor-two problem” [44]. Thus, one may have a black hole temperature which is twice the expected result. One of solutions is proposed by Mitra [45]. Mitra noted that in general, the action I could include some complex constant of integration \mathcal{K} . In this way, the imaginary part of I becomes

$$\text{Im } I_\pm = \text{Im } W_\pm + \text{Im } \mathcal{K} \quad (37)$$

$+/-$ denotes the outgoing/ingoing solutions. In the semi-classical method, the absorption probability and the emission probability for a black hole are given by

$$\begin{aligned} P_{emit} &\propto \exp \left(-\frac{2}{\hbar} \text{Im } I_+ \right) = \exp \left[-\frac{2}{\hbar} (\text{Im } W_+ + \text{Im } \mathcal{K}) \right], \\ P_{abs} &\propto \exp \left(-\frac{2}{\hbar} \text{Im } I_- \right) = \exp \left[-\frac{2}{\hbar} (\text{Im } W_- + \text{Im } \mathcal{K}) \right]. \end{aligned} \quad (38)$$

On the other hand, it is noted that the classical theory of black holes tells us that an incoming particle is absorbed with the probability equalling to one. Thus, one can choose \mathcal{K} to impose the classical constraint on the absorption probability, which is $\text{Im } \mathcal{K} = -\text{Im } W_-$. So eqn. (38) gives that the probability of a particle tunneling from inside to outside the horizon is

$$P_{emit} \propto \exp \left[-\frac{2}{\hbar} (\text{Im } W_+ - \text{Im } W_-) \right]. \quad (39)$$

Another way to circumvent this problem is considering both the contributions from spatial and temporal parts of the action to the tunneling rates.

Spatial Contribution: As pointed out in [44, 46, 47], $\text{Im } W_{\pm} = \text{Im} \int p_r^{\pm} dr$ used in eqns. (38) were not invariant under canonical transformations. Instead, one should take the closed contour integral $\oint p_r dr = \int p_r^+ dr - \int p_r^- dr$, an invariance under canonical transformations, for the tunneling rates $P_{emit/abs} \propto \exp \left(\pm \frac{1}{\hbar} \text{Im} \oint p_r dr \right)$.

Temporal Contribution: As shown in [47–49], the temporal part contribution came from the "rotation" which connects the interior region and the exterior region of the black hole. It was found in [49] that the direction in which the horizon was crossed did not affect the sign of the temporal contribution. However, the sign of the spatial contribution changed when the direction was reversed. Thus, the temporal contributions to $P_{emit/abs}$ were the same. When the horizon was crossed once, the action I got a contribution of $\text{Im}(\omega \Delta t) = \frac{\pi \omega}{2\kappa}$, and for a round trip, the total contribution was $\text{Im}(\omega \Delta t) = \frac{\pi \omega}{\kappa}$.

Taking into account the spatial and temporal contributions, one has for the absorption probability

$$P_{abs} \propto \exp \left[-\frac{1}{\hbar} \left(-\text{Im} \oint p_r dr + \text{Im}(\omega \Delta t) + \text{Im} \mathcal{K} \right) \right], \quad (40)$$

and for the emission probability

$$P_{emit} \propto \exp \left[-\frac{1}{\hbar} \left(\text{Im} \oint p_r dr + \text{Im}(\omega \Delta t) + \text{Im} \mathcal{K} \right) \right], \quad (41)$$

where \mathcal{K} is a constant of integration. Imposing the classical constraint on the absorption probability, one gets

$$\text{Im} \mathcal{K} = \text{Im} \oint p_r dr - \text{Im}(\omega \Delta t), \quad (42)$$

and

$$P_{emit} \propto \left[-\frac{2}{\hbar} \left(\text{Im} \oint p_r dr \right) \right] = \exp \left[-\frac{2}{\hbar} (\text{Im} W_+ - \text{Im} W_-) \right]. \quad (43)$$

Both approaches give the same expression for P_{emit} . There is a Boltzmann factor in P_{emit} with an effective temperature. Using eqns. (19) and (25), we find that the effective temperature for a massive and charged particle is

$$T_{eff} = \frac{T_0}{\sqrt{\alpha} (1 + \Delta_{qm})} + \mathcal{O}(m_p^{-3}), \quad (44)$$

and that for a massless and neutral particle is

$$T_{eff} = \frac{T_0}{1 + \Delta}, \quad (45)$$

where we define $T_0 = \frac{\hbar\kappa}{2\pi}$ and take $k_B = 1$.

E. Discussion

In this section, we suppose that outgoing particles tunnel from $r_1 < r_h$ to $r_2 > r_h$ while ingoing particles from r_2 to r_1 . To obtain the imaginary part of I for the tunneling process, we have to give a prescription for evaluating the integrals of $\text{Im } W_{\pm} = \int_{r_1}^{r_2} p_r^{\pm} dr$. Following the Feynman's $i\epsilon$ -prescription[50], we take the contour of the integral to be an infinitesimal semicircle below the pole at $r = r_h$ for outgoing particles. Thus, the integral becomes

$$\int_{r_1}^{r_2} p_r^+ dr = \int_{r_1}^{r_h - \varepsilon} p_r^+ dr + \int_{C_{B,\varepsilon}} p_r^+ dr + \int_{r_h + \varepsilon}^{r_2} p_r^+ dr, \quad (46)$$

where we denote the semicircle centered at $r = r_h$ with the radius of R going from $r_h + R/r_h - R$ to $r_h - R/r_h + R$ in the upper/lower half complex plane by $C_{U/B,R}$. Since the contributions from the ranges $(r_1, r_h - \varepsilon)$ and $(r_h + \varepsilon, r_2)$ are real, the imaginary part of W_+ is

$$\text{Im } W_+ = \text{Im} \int_{C_{B,\varepsilon}} p_r^+ dr. \quad (47)$$

Similarly, one has for ingoing particles

$$\text{Im } W_- = \text{Im} \int_{C_{U,\varepsilon}} p_r^- dr. \quad (48)$$

To get $\text{Im } W_{\pm}$, we expand p_r^{\pm} in powers of $\frac{\omega}{\sqrt{f(r)m_p}}$

$$p_r^{\pm} = \sum_{n \geq 0} \frac{\omega^{n+1} p_n^{\pm}(r)}{f^{\frac{n}{2}+1}(r) m_p^n}, \quad (49)$$

where $p_n^{\pm}(r)$ are some analytic functions of r around $r = r_h$. For the non-dispersive case, only the first term in eqn. (49) appears. Thus, we can Laurent expand $f(r)$ with respect to r at $r = r_h$ to evaluate $\int_{C_{B/U,\varepsilon}} p_r^{\pm} dr$ as $\varepsilon \rightarrow 0$. However, these expansions for p_r^{\pm} in the dispersive models look suspicious on $C_{U/B,\varepsilon}$ as $\varepsilon \rightarrow 0$. In fact, $\frac{\omega}{\sqrt{f(r)m_p}}$ can become larger than 1 if r is close enough to r_h . Thus, we can not trust the expansions for p_r^{\pm} any more on $C_{U/B,\varepsilon}$. Nevertheless, we can assume that the singularity structure of p_r in the dispersive models is the same as that in the non-dispersive case except the order of the pole at $r = r_h$, which means the MDR effects do not introduce branch cuts or new poles for p_r in the upper or lower half complex plane. Note that one may need a complete theory of quantum gravity

to justify the assumption. Now consider the semicircles $C_{U/B,R}$ with large enough R , which lies in the region where the expansion for p_r can be trusted. Under the assumption, there are no poles inside the area enclosed by $(R, r_h - \varepsilon)$, $C_{U/B,\varepsilon}$, $(r_h + \varepsilon, R)$, and $C_{U/B,R}$. Thus, we have

$$\begin{aligned} \text{Im } W_{\pm} &= \text{Im} \int_{r_1}^{r_h - R} p_r^{\pm} dr + \text{Im} \int_{C_{B/U,R}} p_r^{\pm} dr + \text{Im} \int_{r_h + R}^{r_2} p_r^{\pm} dr \\ &= \text{Im} \int_{C_{B/U,R}} p_r^{\pm} dr = \sum_{n \geq 0} \frac{\omega^{n+1}}{m_p^n} \text{Im} \int_{C_{B/U,R}} \frac{p_n^{\pm}(r)}{f^{\frac{n}{2}+1}(r)} dr, \end{aligned} \quad (50)$$

where contributions from the ranges $(r_1, r_h - R)$ and $(r_h + R, r_2)$ are discarded since they are always real. If the radii of the Laurent series of $\frac{p_n^{\pm}(r)}{f^{\frac{n}{2}+1}(r)}$ expanded at $r = r_h$ are larger than R , we can Laurent expand $\frac{p_n^{\pm}(r)}{f^{\frac{n}{2}+1}(r)}$ on $C_{B/U,R}$ and only the coefficients a_{-1} of $(r - r_h)^{-1}$ terms contribute to the imaginary part of the integrals. This justifies the procedure to obtain to obtain eqns. (19) and (25). Since the expansions for p_r^{\pm} can be trusted on $C_{U/B,R}$, $\frac{\omega}{\sqrt{f(r)m_p}} \sim \frac{\omega}{\sqrt{\kappa R m_p}} \lesssim 1$ on $C_{U/B,R}$, which gives $R \gtrsim \frac{\omega^2}{\kappa m_p^2}$. Usually, one has that the radii of the Laurent series of $\frac{p_n^{\pm}(r)}{f^{\frac{n}{2}+1}(r)} \sim \kappa^{-1}$ and hence $\kappa^{-1} \gtrsim R$. For R to exist, one has $\kappa^{-1} \gtrsim \frac{\omega^2}{\kappa m_p^2}$ which leads to $\omega \lesssim m_p$. Note that if Lorentz violating scale Λ is much smaller than m_p , one instead has that $\frac{\omega}{\sqrt{f(r)\Lambda}} \lesssim 1$ on $C_{U/B,R}$ and $\omega \lesssim \Lambda$.

Various theories of quantum gravity, such as string theory, loop quantum gravity and quantum geometry, predict the existence of a minimal length[51–53]. The generalized uncertainty principle(GUP)[54] is a simply way to realize this minimal length. To incorporate the Klein-Gordon/Dirac equation with the GUP, one usually considers the quantization in position representation. In position representation, the operators $\vec{k} = -i\vec{\nabla}$ and $\omega = i\partial_t$ are introduced[55]. One then can express the energy and momentum operators as functions of \vec{k} and ω and obtain the deformed Klein-Gordon/Dirac equations in flat spacetime. Inserting the ansatz $\varphi = \exp(iEt - i\vec{p} \cdot \vec{x})$ in the Klein-Gordon/Dirac equation gives the dispersion relation for E and p in flat spacetime. In [37–41], the deformed Dirac equation was generalized to curved spacetime. The modified Hawking temperatures of various black holes were then derived via the Hamilton-Jacobi method. It turns out that the way of generalizing the Dirac equation to curved spacetime in [37–41] is the same as that in the appendix of the paper where we choose the static frame as the preferred one. Thus, we can use the dispersion relation for E and p obtained in flat spacetime and eqn. (20) to reproduce the modified Hawking temperatures of black holes with the metric (6) obtained in [37–41]. In

fact, the dispersion relation in flat spacetime in [37–41] is given by

$$p^2 \approx E^2 - m^2 + 2\beta E^4 + \mathcal{O}(\beta^2), \quad (51)$$

which by comparing to eqn. (5) gives

$$\alpha = 1, \beta = 1, \gamma = 0, C_3 = 0, C_4 = 2, \text{ and } m_p = \frac{1}{\sqrt{\beta}}. \quad (52)$$

Thus, eqn. (20) becomes

$$\Delta_{qm} = \beta \left[\frac{24\tilde{\omega}'(r_h)\tilde{\omega}(r_h)}{16\kappa} + \frac{1}{2} \left(\frac{\lambda}{C(r_h^2)} + m^2 \right) - \frac{f''(r_h)\tilde{\omega}^2(r_h)}{4\kappa^2} \right]. \quad (53)$$

For example, Δ_{qm} for a particle with the angular momentum $l = 0$ in a Schwarzschild black hole with the mass M [37, 41] is

$$\Delta_{qm} = \frac{\beta}{2} (m^2 + 4\omega^2). \quad (54)$$

Δ_{qm} for a neutral particle with the angular momentum $l = 0$ in a Reissner-Nordstrom with the mass M and the charge Q [38, 41] is

$$\Delta_{qm} = \beta \left[\frac{m^2}{2} + \frac{2r_+(r_+ - 2r_-)}{(r_+ - r_-)^2} \omega^2 \right], \quad (55)$$

where $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$. Here we only consider a neutral particle to make a comparison since the electromagnetic field was included in [38, 41] in a different way.

Following the argument proposed in [56], the authors in [57] obtained modified relations between the mass of a 4D Schwarzschild black hole and its entropy and temperature. The argument connecting a MDR and some modifications of the entropy of black holes is formulated in a scheme of analysis first introduced by Bekenstein[58]. In fact, the modified temperature of the black hole for the MDR in eqn. (5) with $m = 0$ was given by

$$T = T_0 \left[1 - \frac{m_p C_3}{4M} + \frac{m_p^2}{4M^2} \left(\frac{7C_3^2}{8} - \frac{C_4}{2} \right) + \mathcal{O} \left(\frac{m_p^3}{M^3} \right) \right], \quad (56)$$

where M is the mass of the black hole and $T_0 = \frac{m_p^2}{8\pi M}$. On the other hand, we can use eqn. (20) to estimate the temperature of the black hole. For a massless particle in a 4D Schwarzschild black hole, eqn. (20) gives

$$\Delta = \frac{1}{32m_p^2} \left[(4C_4 - 3C_3^2) \frac{\lambda}{2M^2} + 8(4C_4 - C_3^2) \omega^2 \right] + \mathcal{O}(m_p^{-3}), \quad (57)$$

where λ is the magnitude of the angular momentum of the particle. The event horizon of the Schwarzschild black hole is $r_h = 2M$. Near the horizon of the the black hole, one has $\lambda \sim (pr_h)^2 \sim (\omega r_h)^2$. Thus, one can rewrite Δ

$$\Delta \sim \frac{\omega^2}{16m_p^2} (20C_4 - 7C_3^2) + \mathcal{O}(m_p^{-3}). \quad (58)$$

As reported in [57], the authors obtained the relation $\omega \gtrsim \frac{\hbar}{\delta x} + \mathcal{O}\left(\frac{1}{m_p}\right)$ between the energy of a particle and its position uncertainty for a MDR. Near the horizon of the Schwarzschild black hole, the position uncertainty of a particle is of the order of the Schwarzschild radius of the black hole[58] $\delta x \sim r_h = 2M$. Thus, one finds for T

$$T \sim T_0 \left[1 + \frac{m_p^2}{64M^2} (7C_3^2 - 20C_4) + \mathcal{O}\left(\frac{m_p^3}{M^3}\right) \right]. \quad (59)$$

Note that the term $-\frac{m_p C_3}{4M}$ in eqn. (56) could imply that the leading correction to the entropy of the black hole should have \sqrt{area} dependence. Since in Loop Quantum Gravity such a \sqrt{area} contribution to black-hole entropy has already been excluded, C_3 should vanish[56]. However, it has been shown in [57] that this might be evaded by combining MDR and GUP. Our result suggests that a nonvanishing C_3 in the MDR does not lead to the presence of \sqrt{area} contribution to the black hole entropy.

IV. THERMODYNAMICS OF RADIATIONS

For particles emitted in a wave mode labelled by energy ω and λ plus some other quantum numbers J_i if needed, we find that

$$\begin{aligned} & (\text{Probability for a black hole to emit a particle in this mode}) \\ &= \exp\left(-\frac{\omega}{T_{eff}}\right) \times (\text{Probability for a black hole to absorb a particle in the same mode}), \end{aligned}$$

where T_{eff} is given by eqns. (44) or (45). The above relation for usual dispersion relation was obtained by Hartle and Hawking[59] using semiclassical analysis. If the black hole is in equilibrium, the rate of emission particles by the black hole must exactly equal the rate of absorption. Neglecting back-reaction, detailed balance condition requires that the ratio of the probability of having N particles in a particular mode with ω, λ and J_i to the probability of having $N-1$ particles in the same mode is $\exp\left(-\frac{\omega}{T_{eff}}\right)$. Thus, we find that the probability

of having N particles $P_N(\omega, \lambda, J_i)$ in the mode is given by

$$P_N(\omega, \lambda, J_i) = C_{\omega, \lambda, J_i} \exp\left(-\frac{N\omega}{T_{eff}}\right), \quad (60)$$

where C_{ω, λ, J_i} is a normalizing constant. C_{ω, λ, J_i} is determined by the normalized condition $\sum_{N=0}^{N_\infty} P_N(\omega, \lambda, J_i) = 1$ where $N_\infty = \infty$ for bosons and $N_\infty = 1$ for fermions. Thus, the probability $P_N(\omega, \lambda, J_i)$ is

$$P_N(\omega, \lambda, J_i) = \left[1 - (-1)^\epsilon \exp\left(-\frac{\omega}{T_{eff}}\right)\right]^{1-2\epsilon} \exp\left(-\frac{N\omega}{T_{eff}}\right), \quad (61)$$

where $\epsilon = 0$ for bosons and $\epsilon = 1$ for fermions. To calculate the average number n_{ω, λ, J_i} in the mode, we define

$$A_{\omega, \lambda, J_i}(\mu) = \sum_{N=0}^{N_\infty} \exp\left(N\mu - \frac{N\omega}{T_{eff}}\right), \quad (62)$$

where one has $C_{\omega, \lambda, J_i} = A_{\omega, \lambda, J_i}^{-1}(0)$. So we find

$$n_{\omega, \lambda, J_i} = \sum_{N=0}^{N_\infty} N P_N(\omega, \lambda, J_i) = \frac{\partial_\mu A_{\omega, \lambda, J_i}(\mu)|_{\mu=0}}{A_{\omega, \lambda, J_i}(\mu)} = \frac{1}{\exp\left(\frac{\omega}{T_{eff}}\right) - (-1)^\epsilon}. \quad (63)$$

Using eqns. (61) and (63), one can rewrite $P_N(\omega, \lambda, J_i)$ in terms of n_{ω, λ, J_i} as

$$P_N(\omega, \lambda, J_i) = n_{\omega, \lambda, J_i}^N [1 + (-1)^\epsilon n_{\omega, \lambda, J_i}]^{-N-(-1)^\epsilon}, \quad (64)$$

where N can be any non-negative integer for bosons ($\epsilon = 0$) but is restricted to be 0 or 1 for fermions ($\epsilon = 1$). The von Neumann entropy for the mode is

$$\begin{aligned} s_{\omega, \lambda, J_i} &= - \sum_{N=0}^{N_\infty} P_N(\omega, \lambda, J_i) \ln P_N(\omega, \lambda, J_i), \\ &= [n_{\omega, \lambda, J_i} + (-1)^\epsilon] \ln [1 + (-1)^\epsilon n_{\omega, \lambda, J_i}] - n_{\omega, \lambda, J_i} \ln n_{\omega, \lambda, J_i} \end{aligned} \quad (65)$$

where we use $\sum_{N=0}^{N_\infty} N P_N(\omega, \lambda, J_i) = n_{\omega, \lambda, J_i}$. The total entropy of radiation is

$$S = \sum_{\omega, \lambda, J_i} s_{\omega, \lambda, J_i}, \quad (66)$$

which will be calculated in the brick wall model in section V. Note that since T_{eff} only depends on ω and λ , the average number n_{ω, λ, J_i} and the entropy s_{ω, λ, J_i} are independent of

J_i . Thus, we could omit the subscript J_i in n_{ω,λ,J_i} and s_{ω,λ,J_i} from now on. Defining $n(x)$ and $s(x)$ by

$$\begin{aligned} n(x) &= \frac{1}{\exp x - (-1)^\epsilon}, \\ s(x) &= \frac{(-1)^\epsilon \exp x}{\exp x - (-1)^\epsilon} \ln \left[\frac{\exp x}{\exp x - (-1)^\epsilon} \right] + \frac{\ln [\exp x - (-1)^\epsilon]}{\exp x - (-1)^\epsilon}, \end{aligned} \quad (67)$$

we can write $n_{\omega,\lambda}$ and $s_{\omega,\lambda}$ with respect to $n(x)$ and $s(x)$

$$\begin{aligned} n_{\omega,\lambda} &= n \left(\frac{\omega}{T_{eff}} \right), \\ s_{\omega,\lambda} &= s \left(\frac{\omega}{T_{eff}} \right). \end{aligned} \quad (68)$$

When integrating over ω , we need to specify the upper limit on the energy of the emitted particle. One of the limits comes from the requirement that the energy of the particle could not exceed the mass of the black hole. Another one is from the effective field theories in the appendix. Suppose that the higher dimensional operators in the effective field theories are suppressed by some scale of Lorentz violation Λ . Usually, we can only trust the effective theories below Λ . As decoupling theorem[60] shows, the contributions above Λ in some regularized theory gets absorbed into Wilson coefficients of the effective theories, C s and B s in the appendix. Consequently, the energy of the particle could not exceed Λ otherwise our effective theories would break down. Note that $\omega \lesssim \Lambda$ has also been obtained in section III. Thus, the energy of the emitted particle $\omega \leq \omega_{\max} \equiv \min \{M, \Lambda\}$. In the remaining of the paper, we would encounter the integrals like

$$\int_0^{u_{\max}} u^i n(u) du \text{ or } \int_0^{u_{\max}} u^i s(u) du, \quad (69)$$

where $u_{\max} = \frac{\omega_{\max}}{T_0}$ and i is a non-negative integer. For a black hole with the mass $M \gg m_p$, one finds that $u_{\max} = \frac{2\pi m_p}{m_p^2 \kappa} \sim \frac{1}{\kappa m_p} \gg 1$. For example, $\kappa = \frac{1}{4M}$ and $\kappa m_p \ll 1$ in the Schwarzschild metric. For such case, using $n(x) \sim e^{-x}$ and $s(x) \sim x e^{-x}$ for $x \gg 1$, one gets

$$\int_{u_{\max}}^{\infty} u^i n(u) du \sim \frac{e^{-\frac{1}{\kappa m_p}}}{(\kappa m_p)^i} \text{ and } \int_{u_{\max}}^{\infty} u^i s(u) du \sim \frac{e^{-\frac{1}{\kappa m_p}}}{(\kappa m_p)^{i+1}}, \quad (70)$$

which can be safely neglected for $\kappa m_p \ll 1$ and hence we can let $u_{\max} = \infty$ in eqn. (69). Therefore, in section V, we let $u_{\max} = \infty$ for integrals of the form in eqn. (69) since we are only interested in the divergent part of the entanglement entropy as $\kappa m_p \rightarrow 0$.

Most works of the dispersive models focus on calculating the modifications of the asymptotic spectrum. There are two types of contributions to the asymptotic spectrum measured at infinity: particles generated at the black hole horizon and scattering off the background. The first contribution depends on derivatives of $f(r)$ at $r = r_h$ whereas the second one depends on $f(r)$ for $r > r_h$. As shown in section VI, the spectrum is proportional to $n_{\omega,\lambda}$. Since the tunneling probability across the horizon is calculated in the Hamilton-Jacobi method, only the first contribution to the asymptotic spectrum is computed in our paper. The dispersive models have been studied in 2D spacetime for most works. The higher order terms in the MDR violate conformal invariance of 2D spacetime, hence there is some scattering. Our calculations show that the spectrum of created particles near the horizon is close to a perfect thermal spectrum in the dispersive models with the static preferred frame, where it has been suggested that the asymptotic spectrum could significantly differ from the thermal one[11]. Thus, one might need the scattering off the background to explain the difference between the spectrum near the horizon and the asymptotic one. The near horizon properties of the dispersive modes has been considered in [16]. It was found that the mode mixing responsible for the Hawking effect occurred within a region of size $d_{br} \sim m_p^{\frac{2}{3}} \kappa^{-\frac{1}{3}}$ across the horizon and the standard Hawking spectrum has been recovered as long as $d_{br} \ll \kappa^{-1}$.

In section V, we calculate the thermal entropy near the horizon, where the effects of scattering off the background hardly play a role and hence are neglected. However, the black holes' luminosities at infinity are computed in section VI, where the effects of scattering off the background should be included. In fact, we use the geometric optics approximation to estimate these effects instead of solving the equations of motion of the fields. Under the geometric optics approximation, we find that the asymptotic spectrum is still close to a thermal one. Nevertheless, [11] might suggest that such approximation may not be robust enough and hence the asymptotic spectrum could be significantly changed.

V. ENTROPY IN BRICK WALL MODEL

In 1985 t' Hooft [61] proposed the brick wall model to calculate the entropy of a thermal gas of Hawking particles propagating just outside the black hole horizon. The entropy is calculated by methods of the WKB approximation. However, when it comes to calculate

the density of states of emitted particles, t' Hooft found that they became infinite as one got closer to the horizon. To make the entropy finite, he introduced a brick wall cut-off near the horizon such that the boundary condition

$$\Phi(x) = 0 \quad \text{at } r = r_h + r_\varepsilon, \quad (71)$$

where Φ is the radiation's field. Moreover, another cut-off at a large distance from the horizon $L \gg r_h$ was introduced to eliminate infrared divergences.

For simplicity, we consider in this section a 4D spherically symmetric black hole with the metric

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - C(r^2) d\Omega. \quad (72)$$

For such a black hole, the quantum numbers needed to specify a wave mode of radiation are the energy ω , the angular momentum l , the magnetic quantum number m . We also assume that the radiated particles are massless and neutral. Thus, the MDR, eqn. (2), becomes

$$p^2 = m_p^2 H\left(\frac{E}{m_p}, 0\right) \equiv m_p^2 H\left(\frac{E}{m_p}\right), \quad (73)$$

where we define $H(x) = H(x, 0)$ and $H(x) \sim x^2$ for $x \ll 1$. As shown in section II, the deformed Hamilton-Jacobi equation incorporating eqn. (73) for a massless and neutral particle in the 4D spherically symmetric black hole is given by

$$X^2 = m_p^2 H\left(\frac{T}{m_p}\right), \quad (74)$$

where

$$T = \frac{\omega}{\sqrt{f(r)}}, X^2 = f(r) p_r^2 + \frac{(l + \frac{1}{2})^2 \hbar^2}{C(r^2)}. \quad (75)$$

Then, we get from eqn. (74)

$$p_r^2 = \frac{1}{f(r)} \left[m_p^2 H\left(\frac{\omega}{m_p \sqrt{f(r)}}\right) - \frac{(l + \frac{1}{2})^2 \hbar^2}{C(r^2)} \right]. \quad (76)$$

Define the radial wave number $k(r, l, \omega)$ by

$$k(r, l, \omega) = |p_r|, \quad (77)$$

as long as $p_r^2 \geq 0$, and $k(r, l, \omega) = 0$ otherwise. Taking two Dirichlet conditions at $r = r_h + r_\varepsilon$ and $r = L$ into account, one finds that the number of one-particle states not exceeding ω with fixed value of the angular momentum l is given by

$$n(\omega, l) = \frac{1}{\pi \hbar} \int_{r_h + r_\varepsilon}^L k(r, l, \omega) dr. \quad (78)$$

Thus, we obtain for the total entropy of radiation

$$\begin{aligned}
S &= \sum_{\omega, l, m} s_{\omega, l} = \int (2l+1) dl \int d\omega \frac{dn(\omega, l)}{d\omega} s_{\omega, l} \\
&= \frac{1}{\pi \hbar} \int (2l+1) dl \int d\omega \int_{r_h+r_\varepsilon}^L dr \frac{dk(r, l, \omega)}{d\omega} s_{\omega, l} \\
&= \frac{C(r_h^2) m_p^2}{2\pi \hbar^3} \int_{r_h+r_\varepsilon}^L \frac{dr}{f(r)} \int d\omega H' \left(\frac{\omega}{m \sqrt{f(r)}} \right) \\
&\quad \int_0^{H\left(\frac{\omega}{m_p \sqrt{f(r)}}\right) \frac{C(r_h^2)}{C(r^2)}} dz \left[H \left(\frac{\omega}{m_p \sqrt{f(r)}} \right) - z \frac{C(r_h^2)}{C(r^2)} \right]^{-\frac{1}{2}} s_{\omega, l}, \tag{79}
\end{aligned}$$

where we define a dimensionless parameter $z = \frac{(l+\frac{1}{2})^2 \hbar^2}{C(r_h^2) m_p^2}$. Using $\lambda = (l+\frac{1}{2})^2 \hbar^2$ and $z = \frac{(l+\frac{1}{2})^2 \hbar^2}{C(r_h^2) m_p^2}$, one rewrites eqn. (26) as

$$\Delta = \sum_{a=0}^{\infty} \sum_{k=0}^{\infty} \eta_a^{2k+2a} \zeta_k^a z^a \frac{\omega^{2k}}{m_p^{2k}} - 1. \tag{80}$$

Defining the coefficients $\xi_{l,k}^n$ by

$$\left(\sum_{l'=0}^{\infty} \sum_{k'=0}^{\infty} \eta_{l'}^{2k'+2l'} \zeta_{k'}^{l'} z^{l'} x^{2k'} - 1 \right)^n = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \xi_{l,k}^n z^l x^{2k}, \tag{81}$$

one has for $s_{\omega, l}$

$$\begin{aligned}
s_{\omega, l} &= s \left(\frac{\omega}{T_{eff}} \right) = s \left(\frac{\omega (1 + \Delta)}{T_0} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} s^{(n)} \left(\frac{\omega}{T_0} \right) \frac{\omega^n \Delta^n}{T_0^n} \\
&= \sum_{a=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{s^{(n)}(u) u^n}{n!} \right] \xi_{a,k}^n z^a \left(\frac{T_0 u}{m_p} \right)^{2k} \\
&= \sum_{a=0}^{\infty} z^a \Theta_a(u), \tag{82}
\end{aligned}$$

where we define $u = \frac{\omega}{T_0}$ and

$$\Theta_a(u) = \sum_{k=0}^{\infty} \left(\frac{T_0 u}{m_p} \right)^{2k} \sum_{n=0}^{k+a} \left[\frac{s^{(n)}(u) u^n}{n!} \right] \xi_{a,k}^n. \tag{84}$$

Note that $\xi_{k,l}^0 = 0$ except $\xi_{0,0}^0 = 1$, $\xi_{l,k < n-l}^n = 0$ and $\xi_{l < n-k, k}^n = 0$. Putting eqn. (82) into eqn.

(79) and integrating eqn. (79) over z gives

$$S = \frac{C(r_h^2) m_p^2 T_0}{2\pi \hbar^3} \sum_{a=0}^{\infty} \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2})} \int_0^{\infty} \Theta_a(u) du \int_{r_h+r_\varepsilon}^L \frac{dr}{f(r)} \frac{C^{a+1}(r^2)}{C^{a+1}(r_h^2)} H^{a+\frac{1}{2}} \left(\frac{T_0 u}{m_p \sqrt{f(r)}} \right) H' \left(\frac{T_0 u}{m_p \sqrt{f(r)}} \right). \quad (85)$$

To calculate S , the variable r may be changed by introducing $x = \frac{T_0 u}{m_p \sqrt{f(r)}}$. Then, eqn. (85) becomes

$$S = \frac{C(r_h^2) m_p^2}{4\pi^2 \hbar^2} \sum_{a=0}^{\infty} \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2})} \int_0^{\infty} \Theta_a(u) du \int_{\delta}^{x_\varepsilon} x^{-1} H^{a+\frac{1}{2}}(x) H'(x) G_a \left(\frac{u^2 T_0^2}{x^2 m_p^2} \right) dx, \quad (86)$$

where we define $x_\varepsilon = \frac{T_0 u}{m_p \sqrt{f(r_h+r_\varepsilon)}}$, $\delta = \frac{T_0 u}{m_p \sqrt{f(L)}}$ and

$$G_a(y) = \frac{2\kappa C^{a+1} [f^{-1}(y)^2]}{C^{a+1}(r_h^2) f' [f^{-1}(y)]}. \quad (87)$$

Now the brick walls are at $x = x_\varepsilon$ and $x = \delta$. Note that $x_\varepsilon \rightarrow \infty$ when $r_\varepsilon \rightarrow 0$ and the horizon is at $x_\varepsilon = \infty$. Since $G_a(0) = 1$, we can Taylor expand $G_a(y)$ at $y = 0$

$$G_a(y) = \sum_{k=0}^{\infty} f_k^a y^k, \quad (88)$$

where we find the first two coefficients of the series expansion are $f_0^a = 1$ and $f_1^a = \frac{r_h C'(r_h^2)}{\kappa C(r_h^2)} \left[(a+1) - \frac{C(r_h^2) f''(r_h)}{4\kappa r_h C'(r_h^2)} \right]$. Substituting eqn. (88) into eqn. (86) gives us

$$S = \frac{C(r_h^2)}{4\pi^2 m_p^2} \sum_{k=0}^{\infty} \left(\frac{m_p \kappa}{2\pi} \right)^{2k} \sum_{a=0}^{\infty} \frac{\sqrt{\pi} a! f_k^a}{\Gamma(a + \frac{3}{2})} \int_0^{\infty} u^{2k} \Theta_a(u) du \int_{\delta}^{x_\varepsilon} H^{a+\frac{1}{2}}(x) H'(x) x^{-2k-1} dx, \quad (89)$$

where we use $T_0 = \frac{\hbar \kappa}{2\pi}$ and $\hbar = m_p^2$. The entropy receives two contributions, one from the horizon and the other from the vacuum surrounding the system at large distances. The second one is irrelevant for our purposes and henceforth discarded.

For the usual scenario with $H(x) = x^2$, the integrals over x in eqn. (89) become divergent for $a = k = 0$ as one approaches the horizon with $x_\varepsilon \rightarrow \infty$. This divergence leads to the introduction of the wall near the horizon by t' Hooft. However, the x -integrals could be finite as $x_\varepsilon \rightarrow \infty$ for some MDRs. In fact, there are two kinds of MDRs for the integrals to be finite. For the first kind of these MDRs, the high energy contributions are suppressed. For example, the “all-order MDR” of form

$$p^2 = 2m_p^2 \exp \left(-\frac{E}{m_p} \right) \left[\cosh \left(\frac{E}{m_p} \right) - 1 \right], \quad (90)$$

was given in the κ -Minkowski noncommutative spacetime in [62]. For such a MDR, one has

$$H(x) = 2 \exp(-x) [\cosh(x) - 1],$$

$$H^{l+\frac{1}{2}}(x) H'(x) x^{-2k-1} \sim \exp(-x) x^{-2k-1} \text{ as } x \rightarrow \infty, \quad (91)$$

which guaranties the convergence of the x -integrals as $x_\varepsilon \rightarrow \infty$. Another example is inspired by the all order generalized uncertainty relation considered in [63]. The MDR can be written as

$$\frac{dp}{dE} = \exp\left(-\frac{E^2}{m_p^2}\right), \quad (92)$$

which gives

$$H(x) = \left(\int_0^x e^{-x^2} dx\right)^2,$$

$$H^{l+\frac{1}{2}}(x) H'(x) x^{-2k-1} \lesssim \left(e^{-x^2}\right)^{2l+3} x^{-2k-1} \text{ as } x \rightarrow \infty. \quad (93)$$

Thus, the x -integrals stay finite as $x \rightarrow \infty$. Moreover, it was found in [63] that the entropy kept finite when the wall approached the horizon and hence the wall in the brick wall model located just outside the horizon could be avoided. For the second kind of the MDRs, the energy E in the MDRs has a maximum value and hence x_ε can not go to the infinity. For example, Corley and Jacobson[9] proposed

$$E = \sqrt{p^2 - \frac{p^4}{4m_p^2}}, \quad (94)$$

which gives

$$H(x) = 2(1 - \sqrt{1-x}) \text{ for } 0 \leq x \leq 1,$$

$$H^{l+\frac{1}{2}}(x) H'(x) x^{-2k-1} \sim \frac{1}{\sqrt{1-x}} \text{ as } x \rightarrow 1. \quad (95)$$

So the x -integrals are finite for the Corley and Jacobson dispersion relation. It was shown in [64] that the entropy was rendered UV finite for the Corley and Jacobson dispersion relation. For the Unruh dispersion relation[7]

$$E = m_p \left[\tanh\left(\frac{p^n}{m_p^n}\right) \right]^{\frac{1}{n}}, \quad (96)$$

we have

$$H(x) = [\tanh^{-1}(x^n)]^{\frac{1}{n}} \text{ for } 0 \leq x \leq 1,$$

$$H^{l+\frac{1}{2}}(x) H'(x) x^{-2k-1} \sim \frac{[\ln(1-x^n)]^{(l+\frac{3}{2})\frac{1}{n}-1}}{1-x^n} \text{ as } x \rightarrow 1. \quad (97)$$

We find that the x -integrals diverge as $x \rightarrow 1$ and a wall near the horizon is needed. However, the entropy for the Unruh dispersion relation was also found finite in [64]. This might be due to different generalization of the Hamilton-Jacobi equation in curved space in [64]. In the remaining of section, we will consider two cases, in one of which the x -integrals converge, and in the other they diverge.

A. UV Finite Case

We here assume that the x -integrals converge as $x \rightarrow \Lambda$, where $\Lambda = \infty$ for the first kind of the MDRs in this case and Λ is the largest x for the second kind. Thus, we can define

$$\tilde{c}_k^a = \int_{\delta}^{\Lambda} H^{a+\frac{1}{2}}(x) H'(x) x^{-2k-1} dx. \quad (98)$$

Since $H(x) \sim x^2$ for $x \ll 1$, the Taylor expansion of $H^{a+\frac{1}{2}}(x) H'(x)$ is given by

$$H^{a+\frac{1}{2}}(x) H'(x) = \sum_{j=0}^{\infty} d_j^a x^{j+2a+2}, \quad (99)$$

where $d_0^a = 2$. Then one gets

$$\begin{aligned} \tilde{c}_k^a &= \int_{x_1}^{\Lambda} H^{a+\frac{1}{2}}(x) H'(x) x^{-2k-1} dx + \int_{\delta}^{x_1} \sum_{j=0}^{\infty} d_j^a x^{j+2a+2} x^{-2k-1} dx \\ &= c_k^a - \sum_{j=0, j \neq 2k-2a-2}^{\infty} \frac{d_j^a \delta^{j+2(a-k)+2}}{j+2(a-k)+2} - \theta(k-a-1) d_{2k-2a-2}^a \ln \delta, \end{aligned} \quad (100)$$

where $\theta(x)$ is the Heaviside step function, $0 < x_1 < \Lambda$ and c_k^a is a constant independent of L . Neglecting terms depending on L , one finds

$$\tilde{c}_k^a \sim c_k^a - \theta(k-a-1) d_{2k-2a-2}^a \ln \frac{m_p \kappa u}{2\pi}. \quad (101)$$

Plugging eqn. (101) into eqn. (89) gives us that the entropy near the horizon can be written of form

$$S = \frac{C(r_h^2)}{4\pi^2 m_p^2} \sum_{k=0}^{\infty} \left(s_k + l_k \ln \frac{m_p \kappa}{2\pi} \right) \left(\frac{m_p \kappa}{2\pi} \right)^{2k}. \quad (102)$$

For $k=0$, we can choose $x_1 = 0$ in eqn. (101) since $\int_0^{\Lambda} H^{a+\frac{1}{2}}(x) H'(x) x^{-1} dx$ is convergent as $x \rightarrow 0$. Hence, one has

$$c_0^a = \int_0^{\infty} H^{a+\frac{1}{2}}(x) H'(x) x^{-1} dx, \quad (103)$$

and

$$s_0 = \sum_{a=0}^{\infty} \frac{c_0^a (\eta_1^2)^a \sqrt{\pi}}{\Gamma(a + \frac{3}{2})} \int_0^{\infty} s^{(l)}(u) u^l du, \quad (104)$$

where $\eta_1^2 = \frac{3C_3^2}{16} - \frac{C_4}{4}$ and we use $\xi_{l,k < n-l}^n = 0$, $\xi_{l,0}^l = (\eta_1^2 \zeta_0^1)^l$, and $\zeta_0^1 = 1$. Since there is no $\ln \delta$ in eqn. (101) for $k = 0$, we have $l_0 = 0$. For $k = 1$, only \tilde{c}_1^0 contributes to l_1 and we find

$$l_1 = -4f_1^0 \int_0^{\infty} s(u) u^2 du, \quad (105)$$

where we have

$$f_1^0 = \frac{r_h C'(r_h^2)}{\kappa C(r_h^2)} \left[1 - \frac{C(r_h^2) f''(r_h)}{4\kappa r_h C'(r_h^2)} \right].$$

Thus, we obtains for the entropy near horizon

$$S \sim \frac{C(r_h^2)}{4\pi^2 m_p^2} s_0 - \frac{\kappa r_h C'(r_h^2)}{4\pi^4} \left[1 - \frac{C(r_h^2) f''(r_h)}{4\kappa r_h C'(r_h^2)} \right] \int_0^{\infty} s(u) u^2 du \ln m_p \kappa + \text{Finite terms as } \kappa m_p \rightarrow 0. \quad (106)$$

B. Perturbative Case

In this case, a wall near the horizon is needed to regulate the x -integrals. As above, the function $H(x)$ can be presented in the form of Taylor series

$$H(x) = \sum_{i=2}^{\infty} C_i x^i, \quad (107)$$

where $C_2 = 1$. One then can have Taylor expansions for $H^{a+\frac{1}{2}}(x) H'(x)$

$$H^{a+\frac{1}{2}}(x) H'(x) = \sum_{j=0}^{\infty} d_j^a x^{j+2a+2}, \quad (108)$$

where $d_0^a = 2$. The radial position of the brick wall near the horizon is $r = r_h + r_\varepsilon$ ($x = x_\varepsilon$).

The invariant distance of the wall from the horizon ε is defined by

$$\varepsilon = \int_{r_h}^{r_h+r_\varepsilon} \frac{dr}{\sqrt{f(r)}} = \int_0^{y_\varepsilon} \frac{2dy}{f'(f^{-1}(y^2))}, \quad (109)$$

where we define $y = \frac{T_0 u}{x m_p} = \frac{\sqrt{f(r)}}{u}$ and $y_\varepsilon = \frac{T_0 u}{x_\varepsilon m_p}$. Noting $\frac{f'(f^{-1}(0))}{2\kappa} = 1$, one obtains

$$\varepsilon \kappa = \int_0^{y_\varepsilon} \frac{dy}{\frac{f'(f^{-1}(y^2))}{2\kappa}} = \int_0^{y_\varepsilon} \left(1 + \sum_{n=1}^{\infty} \tilde{f}_n y^{2n} \right) dy = y_\varepsilon \left(1 + \sum_{n=1}^{\infty} \frac{\tilde{f}_n y_\varepsilon^{2n}}{2n+1} \right), \quad (110)$$

where we expand $\frac{2\kappa}{f'(f^{-1}(y^2))}$ in the integral and \tilde{f}_n are coefficients of the series. Solving eqn. (110) for y_ε gives

$$y_\varepsilon = \varepsilon \kappa \left(1 + \sum_{n=1}^{\infty} \zeta_n (\varepsilon \kappa)^{2n} \right), \quad (111)$$

where ζ_n are determined by \tilde{f}_n . Using $x_\varepsilon = \frac{T_0 u}{y_\varepsilon m_p}$, one can relate x_ε to ε by

$$\begin{aligned} x_\varepsilon^a &= \frac{T_0^a u^a}{m_p^a} \frac{1}{(\varepsilon \kappa)^a} \sum_{n=0}^{\infty} \chi_n^a (\varepsilon \kappa)^{2n}, \\ \ln \frac{m_p x_\varepsilon}{T_0 u} &= \sum_{n=1}^{\infty} \chi_0^n (\varepsilon \kappa)^{2n} - \ln \varepsilon \kappa, \end{aligned} \quad (112)$$

where $\chi_a^0 = 1$. Focusing only on the near horizon contributions, we neglect terms involving L and use eqn. (108) to obtain

$$\int_{\delta}^{x_\varepsilon} H^{a+\frac{1}{2}}(x) H'(x) x^{-2k-1} dx \sim \sum_{j=0, j \neq 2k-2a-2}^{\infty} \frac{d_j^a x_\varepsilon^{j+2(a-k)+2}}{j+2(a-k)+2} + \theta(k-a-1) d_{2k-2a-2}^a \ln \frac{m_p x_\varepsilon}{T_0 u}, \quad (113)$$

where for the logarithmic term we have $\int_{\delta}^{x_\varepsilon} x^{-1} dx = \ln \frac{x_\varepsilon}{\delta} = \ln \frac{x_\varepsilon m_p}{T_0 u} - \frac{1}{2} \ln f(L) \sim \ln \frac{x_\varepsilon m_p}{T_0 u}$. Plugging eqns. (113) and (84) into eqn. (89), one finds for the entropy near the horizon

$$\begin{aligned} S &\sim \frac{C(r_h^2)}{16\pi^4 \varepsilon^2} \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2}) (2\pi)^{j+2a}} \left(\frac{m_p}{\varepsilon} \right)^{j+2a} \\ &\quad \sum_{k=0, k \neq \frac{j}{2} + a + 1}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{d_j^a f_k^a \chi_{j+2(a-k)+2}^n}{j+2(a-k)+2} \left(\frac{m_p \kappa}{2\pi} \right)^{2p} (\kappa^2 \varepsilon^2)^{k+n} \sum_{q=0}^{p+a} \frac{\xi_{a,p}^q}{q!} \int_0^\infty u^{j+2l+2p+q+2} s^{(q)}(u) du \\ &\quad + \frac{C(r_h^2) \kappa^2}{16\pi^4} \sum_{n=0}^{\infty} (\kappa^2 \varepsilon^2)^n \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2})} d_{2j}^a f_{j+a+1}^a \chi_0^n \left(\frac{m_p \kappa}{2\pi} \right)^{2j+2a+2p} \\ &\quad \sum_{q=0}^{p+a} \xi_{a,p}^q \int_0^\infty u^{2j+2a+2p+q+2} \frac{s^{(q)}(u)}{q!} du \\ &\quad - \ln(\kappa \varepsilon) \frac{C(r_h^2) \kappa^2}{16\pi^4} \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2})} d_{2j}^a f_{j+a+1}^a \left(\frac{m_p \kappa}{2\pi} \right)^{2j+2a+2p} \\ &\quad \sum_{q=0}^{p+a} \frac{\xi_{a,p}^q}{q!} \int_0^\infty s^{(q)}(u) u^{2j+2a+2p+q+2} du. \end{aligned} \quad (114)$$

At first sight, it seems impossible to single out the most divergent part of eqn. (114) since $j+2a$ in the first term of eqn. (114) can go to infinity. However, the brick wall is put at $r = r_h + r_\varepsilon$ to cut off some unknown quantum physics of gravity. In this sense, the

invariant distance of the wall from the horizon ε could be given by $\varepsilon \sim m_p$. Indeed in the 't Hooft's original calculation for Schwarzschild black holes, requiring that the entropy of the radiation near the horizon $S_{Brick} \lesssim$ the Black hole's Bekenstein-Hawking entropy S_{BH} also gives $\varepsilon \gtrsim \sqrt{\frac{1}{90\pi}} m_p$ for a scalar field. Thus, we define α such as $\varepsilon = \alpha m_p$. Replacing ε by αm_p in eqn. (114), we find for the entropy

$$S \sim \frac{C(r_h^2)}{4\pi^2 m_p^2} s_0 - \frac{\kappa r_h C'(r_h^2)}{4\pi^4} \left[1 - \frac{C(r_h^2) f''(r_h)}{4\kappa r_h C'(r_h^2)} \right] \int_0^\infty s(u) u^2 du \ln \kappa m_p + \text{Finite terms as } m_p \kappa \rightarrow 0, \quad (115)$$

where we define

$$s_0 = \frac{1}{4\pi^2 \alpha^2} \sum_{a=0}^\infty \sum_{j=0}^\infty \frac{\sqrt{\pi} a!}{\Gamma(a + \frac{3}{2})} \left(\frac{1}{2\pi\alpha} \right)^{j+2a} \frac{d_j^a}{j+2a+2} \left(\sum_{q=0}^a \frac{\xi_{a,0}^q}{q!} \int_0^\infty u^{j+2a+q+2} s^{(q)}(u) du \right). \quad (116)$$

C. Discussion

For a massless scalar field, we find the entropy near horizon in both cases can be written as

$$S \sim \frac{A s_0}{16\pi^3 m_p^2} + s_L \ln m_p \kappa + \text{Finite terms as } \kappa m_p \rightarrow 0, \quad (117)$$

where $s_L = -\frac{\kappa r_h C'(r_h^2)}{45} \left[1 - \frac{C(r_h^2) f''(r_h)}{4\kappa r_h C'(r_h^2)} \right]$ and $A = 4\pi C(r_h^2)$ is the horizon area. For the scenario without the MDR, the entropy near horizon [61, 65, 66] is

$$S \sim \frac{A}{360\alpha^2 \pi m_p^2} + s_L \ln m_p \kappa + \text{Finite terms as } \kappa m_p \rightarrow 0, \quad (118)$$

where we let the proper distance $\varepsilon = \alpha m_p$. By comparing eqn. (117) with eqn. (118) it shows that the leading term of the entropy is affected by the effects of the MDR while the subleading logarithmic term is not. On the other hand, the first law of black hole thermodynamics $dS_B = \frac{dM}{T}$ and eqn. (59) lead to the modified entropy of the black hole

$$S_B \sim \frac{A}{4m_p^2} + \frac{\pi}{8} (7C_3^2 - 20C_4) \ln \kappa m_p + \text{Finite terms as } m_p \kappa \rightarrow 0, \quad (119)$$

where $A = 16\pi M^2$ and $\kappa = \frac{1}{4M}$. For S_B , the leading term is not changed while the subleading logarithmic term is due to the MDR. These might suggest that the explanations for statistical origin of the black holes' entropy need more than the entropy of a thermal gas of Hawking particles near the horizon.

Since the deformed Hamilton-Jacobi equations and the corrections to the Hawking temperature are same for fermions and scalars with the same MDR, one may wonder if eqns. (106) and (115) also work for fermions. In fact, it has been shown in [67] that the same argument in this section held for fermions if an appropriate boundary condition was taken instead of the too restrictive Dirichlet boundary condition.

For a MDR with $H(x)$ in the UV finite case, we have shown that a brick wall near the horizon is not needed since the entropy is finite as one approaches the horizon. However, if one expands $H(x)$ as a power series of x and calculates the entropy in the perturbative case, it seems that a wall near the horizon is needed to regulate the divergence. How can we reconcile the contradiction? As noted in [68], the divergence of the entropy in the perturbative case as $\alpha \rightarrow 0$ is more like due to the breaking down of the Taylor series. For the typical energy $\omega \sim T_0 = \frac{\hbar\kappa}{2\pi}$, one finds that $H(x)$ and the MDR corrections to the entropy are powers of $\frac{\omega}{\sqrt{f(r)m_p}} \sim \frac{\hbar\kappa}{2\pi\sqrt{f(r)m_p}}$. At the wall at $r_\epsilon \approx r_h + 2\kappa m_p^2$, we have $\frac{\hbar\kappa}{\sqrt{f(r_\epsilon)m_p}} \sim \frac{1}{4\pi}$. Thus, the perturbative case is valid outside the wall at $r_\epsilon = r_h + 2\kappa m_p^2$. However, the perturbation would break down deep within the wall and the closed form of $H(x)$ is needed.

VI. BLACK HOLE EVAPORATION

In [69], Page counted the number of modes per frequency interval with periodic boundary conditions in a large container around the black hole and divided it by the time it takes a particle to cross the container. He then related the expected number emitted per mode n to the average emission rate per frequency interval $\frac{dn}{dt}$ by

$$\frac{dn}{dt} = n \frac{d\omega}{2\pi\hbar}, \quad (120)$$

for each mode and frequency interval $(\omega, \omega + d\omega)$. Following the same argument, we find that in the MDR case

$$\frac{dn}{dt} = n \frac{\partial\omega}{\partial p_r} \frac{dp_r}{2\pi\hbar} = n \frac{d\omega}{2\pi\hbar}, \quad (121)$$

where $\frac{\partial\omega}{\partial p_r}$ is the radial velocity of the particle and the number of modes between the wavevector interval $(p_r, p_r + dp_r)$ is $\frac{dp_r}{2\pi\hbar}$. Since each particle carries off the energy ω , the total luminosity is obtained from multiplying $\frac{dn}{dt}$ by the energy ω and summing up over all energy ω

and quantum numbers, denoted by i ,

$$L = \sum_i \int \omega n_{\omega,i} \frac{d\omega}{2\pi\hbar}. \quad (122)$$

However, some of the radiation emitted by the horizon might not be able to reach the asymptotic region. Before the radiation reaches the distant observer, they must pass the curved spacetime around the black hole horizon, which plays the role of a potential barrier. This effect on L can be described by a greybody factor from the scattering coefficients of the black hole. Actually, the greybody factor is given by $|T_i(\omega)|^2$, where $T_i(\omega)$ represents the transmission coefficient of the black hole barrier which in general can depend on the energy ω and quantum numbers i of the particle. Taking the greybody factor into account, we find for the total luminosity

$$L = \sum_i \int |T_i(\omega)|^2 \omega n_{\omega,i} \frac{d\omega}{2\pi\hbar}. \quad (123)$$

Since the relevant radiation usually have the energy of order $\hbar M^{-1}$, where M is the mass of the black hole, one should use the wave equations given in the appendix to compute $|T_i(\omega)|^2$ accurately. However, solving the wave equations for $|T_i(\omega)|^2$ could be very complicated. On the other hand, one can use the geometric optics approximation to estimate $|T_i(\omega)|^2$. In the geometric optics approximation, we assume $\omega \gg M$ and high energy waves will be absorbed unless they are aimed away from the black hole. Hence we have $|T_i(\omega)|^2 = 1$ for all the classically allowed energy ω and quantum numbers i of the particle, while $|T_i(E)|^2 = 0$ otherwise. For the usual dispersion relations, the Stefan's law for black holes is obtained in this approximation. In the remaining of the section, we will discuss evaporations of a 4D spherically symmetric black hole with the mass $M \gg m_p$ and a 2D black hole. For simplicity, we assume that the particles are massless and neutral.

A. 4D Spherically Symmetric Black Hole

To find the classically allowed values of angular momentum l with fixed value of energy ω , we consider eqn. (76) for a massless particle in a 4D spherically symmetric black hole, where we have $\lambda = \left(l + \frac{1}{2}\right)^2 \hbar^2$. Since p_r is always a real number in the geometric optics approximation, one has an upper bound on λ

$$\lambda \leq C(r^2) m_p^2 H \left(\frac{\omega}{m_p \sqrt{f(r)}} \right). \quad (124)$$

Suppose $C(r^2)m_p^2 H\left(\frac{\omega}{m_p\sqrt{f(r)}}\right)$ has a minimum at r_{\min} and this minimum is denoted by λ_{\max} . If the particles overcome the angular momentum barrier and get absorbed by the black hole, one must have $\lambda \leq \lambda_{\max}$. Thus, the eqn. (123) becomes

$$\begin{aligned} L &= g_s \int \frac{\omega d\omega}{2\pi\hbar^3} \int_0^{\lambda_{\max}} n\left(\frac{\omega(1+\Delta)}{T_0}\right) d\left[\left(l+\frac{1}{2}\right)^2 \hbar^2\right] \\ &= \frac{g_s C(r_h^2)m_p^2}{2\pi\hbar^3} \int \omega d\omega \int_0^{\frac{C(r_{\min}^2)}{C(r_h^2)} H\left(\frac{\omega}{m_p\sqrt{f(r_{\min})}}\right)} n\left(\frac{\omega(1+\Delta)}{T_0}\right) dz, \end{aligned} \quad (125)$$

where g_s is the number of polarization, $z = \frac{(l+\frac{1}{2})^2 \hbar^2}{C(r_h^2)m_p^2}$ and we use eqn. (68) for $n_{\omega,l}$. Defining $n_a(u)$ by

$$n_a(u) = \sum_{k=0}^{\infty} \left(\frac{T_0 u}{m_p}\right)^{2k} \sum_{n=0}^{k+a} \left[\frac{n^{(n)}(u) u^n}{n!}\right] \xi_{a,k}^n, \quad (126)$$

we find

$$n\left(\frac{\omega(1+\Delta)}{T_0}\right) = \sum_{a=0}^{\infty} z^a n_a(u), \quad (127)$$

where $\xi_{a,k}^n$ is given by eqn. (81). Substituting eqn. (127) into eqn. (125) and integrating eqn. (125) over z gives

$$L = \frac{g_s C(r_h^2) T_0^2}{2\pi\hbar^3} \sum_{a=0}^{\infty} \frac{m_p^2}{(a+1)} \frac{C^{a+1}(r_{\min}^2)}{C^{a+1}(r_h^2)} \int_0^{\infty} u n_a(u) H^{a+1}\left(\frac{T_0 u}{m_p\sqrt{f(r_{\min})}}\right) du, \quad (128)$$

where we let $u_{\max} = \infty$ for $M \gg m_p$. Since $H(x) = x^2 + \sum_{n \geq 3} C_n x^n$, we define h_m^a by

$$H^a(x) = x^{2a} \sum_{m=0}^{\infty} h_m^a x^m, \quad (129)$$

where $h_0^a = 1$, $h_1^a = aC_3$ and $h_2^a = C_4 a + \frac{C_3^2(a-1)a}{2}$. Plugging eqns. (129) and (126) into eqn. (128) gives

$$L = \frac{g_s C(r_h^2) T_0^4}{2\pi\hbar^3} \sum_{j=0}^{\infty} \left(\frac{T_0}{m_p}\right)^j \sum_{a=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{a+1} \left[\sum_{k=0}^{\lfloor \frac{j}{2} \rfloor - a} h_{j-2a-2k}^{a+1} \left(\sum_{i=0}^{k+a} \frac{\xi_{a,k}^i}{i!} N_{i,j,l,k} \right) \right], \quad (130)$$

where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ and we define

$$N_{i,j,a,k} = \int_0^{\infty} \frac{n^{(i)}(u) u^{j+i+3}}{f^{\frac{j}{2}-k+1}(r_{\min})} \frac{C^{a+1}(r_{\min}^2)}{C^{a+1}(r_h^2)} du. \quad (131)$$

We now use eqn. (130) to calculate the luminosity in the Schwarzschild metric to $\mathcal{O}\left(\frac{m_p^2}{M^2}\right)$. For the Schwarzschild metric, one has $f(r) = 1 - \frac{2M}{r}$, $r_h = 2M$, $C(r^2) = r^2$ and $\kappa = \frac{1}{4M}$. Taking the derivative of $C(r^2) m_p^2 H\left(\frac{\omega}{m_p \sqrt{f(r)}}\right)$ and equating it to zero, one finds

$$\begin{aligned} r_{\min} &= 3M \left(1 + \frac{\sqrt{3}C_3}{6} \frac{T_0 u}{m_p} + \frac{12C_4 - 7C_3^2}{12} \frac{T_0^2 u^2}{m_p^2} + \mathcal{O}\left(\frac{T_0^3}{m_p^3}\right) \right), \\ \lambda_{\max} &= 27M^2 u^2 T_0^2 \left(1 + \sqrt{3}C_3 \frac{T_0 u}{m_p} + \frac{12C_4 - C_3^2}{4} \frac{T_0^2 u^2}{m_p^2} + \mathcal{O}\left(\frac{T_0^3}{m_p^3}\right) \right). \end{aligned} \quad (132)$$

For emitting n_s species of massless scalars and n_f species of massless spin-1/2 fermions from a Schwarzschild black hole into empty space, putting eqns. (132) into eqn. (130) gives the total luminosity

$$\begin{aligned} L &= \frac{9m_p^2}{40960\pi M^2} \left\{ \left(n_s + \frac{7}{4}n_f \right) + (0.26n_s + 0.50n_f) C_3 \frac{m_p}{M} \right. \\ &\quad \left. + \left[(0.15n_s + 0.29n_f) C_3^2 - (0.24n_s + 0.46n_f) C_4 \right] \frac{m_p^2}{M^2} + \mathcal{O}\left(\frac{m_p^3}{M^3}\right) \right\}. \end{aligned} \quad (133)$$

In the geometric optics approximation, the Schwarzschild black hole can be described as a black sphere for absorbing particles. The total luminosity are determined by the radius of the black sphere R and the temperature of the black hole T . Note that one has $R = \sqrt{\frac{\lambda_{\max}}{\omega^2}}$ and $T_{eff} \approx T_0 (1 - \Delta)$ where for massless particles

$$\Delta = \frac{1}{32m_p^2} \left[(4C_4 - 3C_3^2) \frac{\lambda}{2M^2} + 8(4C_4 - C_3^2) \omega^2 \right]. \quad (134)$$

Consider a sub-luminal case with $C_3 = 0$ and $C_4 > 0$, where the total luminosity decreases due to the MDR effects. In this case, the MDR effects increase the radius of the black sphere while they decrease the temperature of the black hole. The competition between the increased radius and the decreased temperature determines whether the luminosity would increase or decrease. It appears from eqn. (133) that the effects of decreased temperature wins the competition.

B. 2D Black Hole

Suppose the metric of a 2D black hole is given by

$$ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2, \quad (135)$$

where $f(r)$ has a simple zero at $r = r_h$. Here we consider a neutral and massless scalar particle governed by the modified dispersion relation

$$E^2 = p^2 - \frac{Cp^4}{m_p^2}, \quad (136)$$

which is the Corley and Jacobson dispersion relation for $C > 0$ [9]. Expressing p in terms of E gives

$$p^2 = E^2 - \frac{CE^4}{m_p^2} + \mathcal{O}\left(\frac{E^4}{m_p^4}\right). \quad (137)$$

For the 2D black hole with the event horizon at $r = r_h$, eqn. (20) gives

$$\Delta = -\frac{f''(r_h)}{8\kappa^2} \frac{\omega^2}{m_p^2} C + \mathcal{O}\left(\frac{\omega^4}{m_p^4}\right), \quad (138)$$

where we use $m = 0$, $\tilde{\omega}(r_h) = \omega$, $\lambda = 0$, $\alpha = 1$, $\gamma = 0$, $C_3 = 0$, and $C_4 = C$. For $\omega < \omega_{\max}$, the term $\frac{\omega^2 C}{m_p^2}$ in eqn. (138) dominates and hence the terms $\mathcal{O}\left(\frac{\omega^4}{m_p^4}\right)$ are neglected for simplicity. Define $\eta = -\frac{f''(r_h)}{8\kappa^2}$ which becomes 1 for a 2D Schwarzschild black hole with $f(r) = 1 - \frac{2M}{r}$. In this case, we can choose the cutoff of the effective theories $\Lambda = \frac{\alpha m_p}{\sqrt{|\eta C|}}$, where $0 < \alpha < 1$. Note that $|\Delta| < 1$ for $\omega < \Lambda$. Therefore, the luminosity for the black hole is

$$\begin{aligned} L &= \int_0^{\omega_{\max}} \omega n \left[\frac{\omega}{T_0} \left(1 + \frac{\omega^2 \eta C}{m_p^2} \right) \right] \frac{d\omega}{2\pi\hbar} \\ &= \frac{\kappa^2 m_p^2}{8\pi^3} \int_0^{u_{\max}} u n \left[u \left(1 + \frac{\eta C \kappa^2 m_p^2 u^2}{4\pi^2} \right) \right] du, \end{aligned} \quad (139)$$

where $T_0 = \frac{\hbar\kappa}{2\pi}$ and $u_{\max} = \min \left\{ \frac{2\pi M}{\kappa m_p^2}, \frac{\alpha}{\sqrt{|\eta C|}} \frac{2\pi}{m_p \kappa} \right\}$. For $\kappa m_p \ll 1$, we can let $u_{\max} = \infty$ and then find

$$L = \frac{\kappa^2 m_p^2}{48\pi} \left(1 - \frac{2\eta C}{5} \kappa^2 m_p^2 + \mathcal{O}(\kappa^4 m_p^4) \right). \quad (140)$$

For $M < \frac{\alpha m_p}{\sqrt{|\eta C|}}$, we have $u_{\max} = \frac{2\pi M}{\kappa m_p^2}$ and find

$$L = \frac{\kappa M}{4\pi^2} \left[1 - \frac{\pi \kappa M}{2\kappa^2 m_p^2} \left(1 + \frac{2\eta C \kappa M}{3\pi} \right) + \mathcal{O}(\kappa^{-4} m_p^{-4}) \right]. \quad (141)$$

From eqn. (140) for small κm_p and eqn. (141) for large κm_p , we can conclude that the coefficients C impacts the black hole's luminosity only in the intermediate range of κm_p noticeably.

For the intermediate range, FIG. 1 plots the luminosity L against $(4\kappa m_p)^{-1}$, which becomes $\frac{m_p}{M}$ for a 2D Schwarzschild black hole with the mass M . In FIG. 1, we have $\alpha = 0.9$

and $M = \frac{1}{4\kappa}$ for u_{\max} . We plot L vs $(4\kappa m_p)^{-1}$ in FIG. 1 for the usual case with $C = 0$ (red line), the ones with $\eta C = 10$ and 1000 (solid and dashed blue lines, respectively) and the ones with $\eta C = -10$ and -1000 (solid and dashed brown lines, respectively). For the $\eta C < 0$ cases, there are "weird" peaks in FIG. 1, which are due to the transition from $\frac{2\pi M}{\kappa m_p^2}$ to $\frac{\alpha}{\sqrt{|\eta C|}} \frac{2\pi}{m_p \kappa}$ in u_{\max} . However, such transitions are barely seen for the $\eta C > 0$ cases. In our calculations, the luminosities are determined not only by the modified Hawking temperature but also the range of integration of u in eqn. (139). When $M > \frac{\alpha m_p}{\sqrt{|\eta C|}}$, in the $\eta C < 0 / \eta C > 0$ cases the ranges of integration are less than that in the usual case, which tends to decrease the luminosity. In the $\eta C > 0$ cases, it shows from eqn. (138) that the modified Hawking temperatures are lower than that in the usual case. Thus, the luminosities L become smaller due to the decreased temperature and the shrunk range. From eqn. (138), we find that the modified Hawking temperatures in the $\eta C < 0$ cases are higher than that in the usual case. Thus, the competition between the increased temperature and the shrunk range determines the luminosity. The effect of the increased temperature dominates over that of the shrunk range for $\eta C = -10$ and vice versa for $\eta C = -1000$.

To see how the luminosities L depend on values of α , we plot L vs $(4\kappa m_p)^{-1}$ in FIG. 2 for the usual case with $C = 0$ (red line), the ones with $\eta C = 10$ (blue lines) and the ones with $\eta C = -10$ (brown lines) with $\alpha = 0.5$ (solid lines), 0.9 (dashed lines) and 0.95 (dotted lines). Note that α parameterizes the unknown quantum gravity ultraviolet cutoff Λ . In FIG. 2, it suggests that the $\eta C < 0$ cases are highly sensitive to the physics at high energies while the $\eta C > 0$ ones are not. If $\eta > 0$, $\eta C < 0 / \eta C > 0$ implies that the particles are super-/sub-luminal. The author in [70] has shown that the Hawking radiation with sub-luminal dispersion was not sensitive to Lorentz violation at high energies due to the "mode conversion". However, the outgoing black hole modes with super-luminal dispersion emanated from some unknown quantum gravity processes.

VII. DISCUSSION AND CONCLUSION

We used the Hamilton-Jacobi method to investigate the dispersive field theory models in this paper. Our results suggest that the thermal spectrum of radiations near horizon is robust. In fact, if the difference between the modified dispersion relation and the relativistic one was suppressed by the fundamental energy scale m_p , we found that the deviation of

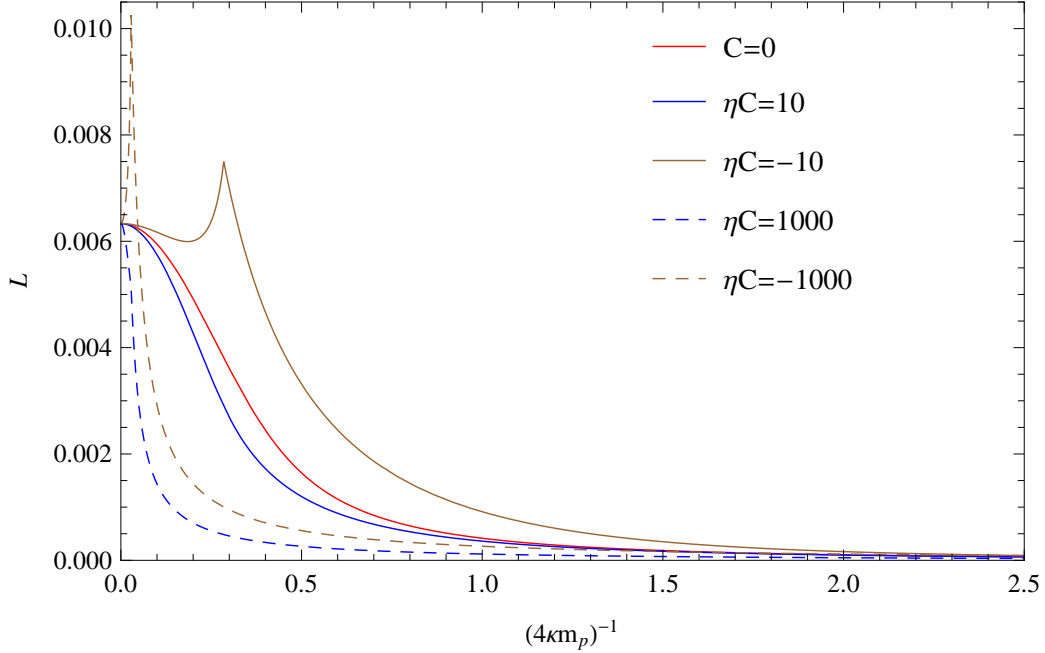


FIG. 1: The luminosity L of a 2D black hole against $(4\kappa m_p)^{-1}$ with $\alpha = 0.9$.

the effective Hawking temperature from the standard one was also suppressed by m_p . For a particle with the typical energy $\omega \sim \frac{m_p^2}{M}$, the deviation was given by powers of $\frac{m_p^2}{M^2}$. Nevertheless, there are some potential corrections to the effective Hawking temperature which are not included in our calculations:

- (a) Back-reaction effects which occurs at order $\frac{\omega}{M}$. For a particle with $\omega \sim \frac{m_p^2}{M}$, they are order of $\frac{m_p^2}{M^2}$. However, the Hamilton-Jacobi method is incapable of computing them since the metric is fixed in this method. On the other hand, back-reaction appears in the the null geodesic method[20, 21] to ensure energy conservation during the emission of a particle via tunneling through the horizon. These corrections lead to non-thermal corrections to the black-hole radiation spectrum. Note that there are some attempts to incorporate back-reaction effects into the the Hamilton-Jacobi method using the rainbow metric[71, 72].
- (b) Higher order WKB corrections. In the Hamilton-Jacobi method, we take the semi-classical limit $\hbar \rightarrow 0$ and keep only leading order terms to calculate the Hawking temperature. Therefore, one may wonder if the Hawking temperature could receive higher order corrections in \hbar beyond the semiclassical one. The corrections has been estimated in [73] and was given by powers of $\frac{m_p^2}{M^2}$. However for the non-dispersive

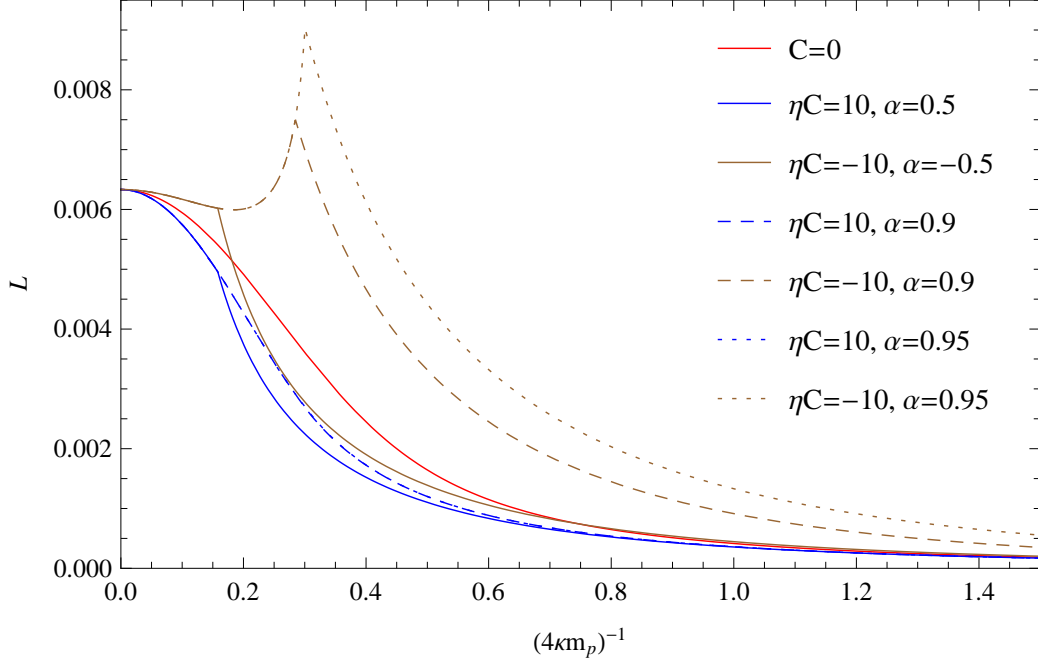


FIG. 2: The luminosity L of a 2D black hole against $(4\kappa m_p)^{-1}$ with $a = 0.5, 0.9$ and 0.95 .

case, several authors[74–76] argued that the tunneling method yielded no higher-order corrections to the Hawking temperature. Whether such arguments also work for the dispersive models needs to be checked.

In this paper, we used the Hamilton-Jacobi method to calculate tunneling rates of radiations across the horizon and the effective Hawking temperatures in the dispersive models with the static preferred frame. After the spectrum of radiations near the horizon was obtained, the thermal entropy of radiations near the horizon and the luminosity of the black hole were computed. Our main results are as follows:

- In section II and the appendix, we used heuristic arguments and effective field theories, respectively to derive the deformed Hamilton-Jacobi equations in the dispersive models with the static preferred frame. Note that these methods can easily be generalized to any preferred frame.
- In section II, the deformed Hamilton-Jacobi equations was solved for $\partial_r I$ and the imaginary part of I was obtained by computing the residue of $\partial_r I$ at $r = r_h$. The assumption for our calculation was also given, which required that the singularity structure of $\partial_r I$ except the order of the pole at $r = r_h$ do not change after the MDR

was introduced. The corrections to the Hawking temperature were calculated for massive and charged particles to $\mathcal{O}(m_p^{-2})$ and neutral and massless particles to all orders, respectively. It was found that corrections were suppressed by m_p .

- In section IV, the average number and entropy for a mode were calculated for bosons and fermions. They could be obtained from those in the non-dispersive case by replacing the standard Hawking temperature with the modified one.
- In section V, we used the brick wall model to compute the thermal entropy of a massless scalar field near the horizon in UV finite and perturbative cases. In the UV finite case, the entropy was always finite as one approached the horizon and hence the wall near the horizon was not needed. In the perturbative case, a wall was put at $r = r_h + r_\epsilon$ to regulate the UV divergence. We assumed the proper distance between the horizon and the wall is order of m_p . Thus, the entropies near the horizon in both cases were given in eqn. (117). We found that the leading term of the entropy depended on the MDR effects while the subleading logarithmic term did not.
- In section VI, we calculated luminosities of a 4D spherically symmetric black hole with the mass $M \gg m_p$ and a 2D one. We used the geometric optics approximation to estimate the effects of scattering off the background. However, as discussed in section IV, such approximation might not be robust enough and hence the asymptotic spectrum could be significantly changed.

Acknowledgments

We are grateful to Houwen Wu and Zheng Sun for useful discussions. This work is supported in part by NSFC (Grant No. 11005016, 11175039 and 11375121) and the Fundamental Research Funds for the Central Universities.

Appendix A: Effective Field Theory and Deformed Hamilton-Jacobi Equation

As discussed in the introduction, various approaches to the quantum-gravity problem could lead to the existence of MDRs. To have a MDR, one has to break or modify the global Lorentz symmetry in the classical limit of the quantum gravity. There are several

possibilities for breaking or modifying the Lorentz symmetry, one of which is that Lorentz invariance is spontaneously broken by extra tensor fields taking on vacuum expectation values. The most conservative approach for a framework in which to describe MDR is the effective field theory (EFT), where modifications to the dispersion relation can be described by the higher dimensional operators. Since we are only interested in modifications to the dispersion relation of the particles, we limit ourselves to the kinetic terms and neglect self-interacting effective operators when constructing the effective field theory. We also assume that the effective theory respects $U(1)$ gauge invariance of the charged black hole. The EFT framework can easily incorporate MDR via the introduction of extra tensors. To construct the minimal EFT in curved spacetime, we suppose that the action of the EFT contains the usual minimal gravitational couplings and the EFT coefficients are constants in the local frame[77].

1. Scalar Field

We work with a complex scalar field ϕ with the mass m and the charge q . Following guidelines we put forth, we find the effective Lagrangian for ϕ incorporating MDR can be written as

$$\begin{aligned} \mathcal{L}_{eff}^s = & -\phi^+ \left(D^\mu D_\mu + \frac{m^2}{\hbar^2} \right) \phi - \frac{m^2 B\left(\frac{m}{m_p}\right)}{\hbar^2} \phi^+ \phi - \frac{imC_\mu\left(\frac{m}{m_p}\right)}{\hbar} \phi^+ D^\mu \phi \\ & - \sum_{n \geq 2, j} \left(\frac{\hbar}{i} \right)^{n-2} \frac{C_{\mu_1 \dots \mu_n}^j\left(\frac{m}{m_p}\right)}{m_p^{n-2}} \phi^+ D^{\mu_1} \dots D^{\mu_n} \phi, \end{aligned} \quad (A1)$$

where $D_\mu = \nabla_\mu + \frac{iq}{\hbar} A_\mu$, ∇_μ is the covariant derivative of the background spacetime, A_μ is the electromagnetic potential, j runs over all independent operators of a given dimension, B is a dimensionless function of $\frac{m}{m_p}$ with $B(0) = 0$, and, C_μ and $C_{\mu_1 \dots \mu_n}^j$ are dimensionless extra tensors depending on $\frac{m}{m_p}$ with $C_\mu(0) = C_{\mu\nu}^j(0) = 0$. The deformed Klein-Gordon equation is

$$\begin{aligned} & - \left(D^\mu D_\mu + \frac{m^2}{\hbar^2} \right) \phi - \frac{m^2 B\left(\frac{m}{m_p}\right)}{\hbar^2} \phi - \frac{imC_\mu\left(\frac{m}{m_p}\right)}{\hbar} D^\mu \phi \\ & - \sum_{n \geq 2, j} \left(\frac{\hbar}{i} \right)^{n-2} \frac{C_{\mu_1 \dots \mu_n}^j\left(\frac{m}{m_p}\right)}{m_p^{n-2}} D^{\mu_1} \dots D^{\mu_n} \phi = 0. \end{aligned} \quad (A2)$$

With rotational symmetry, all extra tensors become reducible to products of a vector field u^μ , which describes the preferred frame and $u^\mu u_\mu = 1$. Thus, the extra tensors become

$$C_\mu = C \left(\frac{m}{m_p} \right) u^\mu, \\ C_{\mu_1 \dots \mu_n}^j = C_n^j \left(\frac{m}{m_p} \right) g_{\mu_{i_1} \mu_{i_2}} \dots g_{\mu_{i_{2k-1}} \mu_{i_{2k}}} u_{\mu_{i_{2k+1}}} \dots u_{\mu_{i_n}}, \quad (\text{A3})$$

where $g_{\mu\nu}$ is the metric of the background spacetime, C and C_n^j are dimensionless functions of $\frac{m}{m_p}$, $j = (k, \mathcal{C})$, $2k \leq n$, and \mathcal{C} denotes any possible permutations of $(1, \dots, n)$, namely (i_1, \dots, i_n) . To obtain the Hamilton-Jacobi equation, we make the WKB ansatz for ϕ

$$\phi = \exp \left(\frac{iI}{\hbar} \right). \quad (\text{A4})$$

Defining

$$\tilde{T} = -u^\mu (\partial_\mu I + qA_\mu), \quad \tilde{X}^2 = \tilde{T}^2 - (\partial_\mu I + qA_\mu)^2, \quad (\text{A5})$$

and plugging eqns. (A4) and (A3) into eqn. (A2), one expands eqn. (A2) in powers of \hbar and finds to the lowest order

$$\left(\tilde{T}^2 - \tilde{X}^2 - m^2 \right) - m^2 B \left(\frac{m}{m_p} \right) - m C \left(\frac{m}{m_p} \right) \tilde{T} + \sum_{n \geq 2, k \leq \frac{n}{2}, \mathcal{C}} \frac{(-1)^n C_n^j \left(\frac{m}{m_p} \right) \left(\tilde{T}^2 - \tilde{X}^2 \right)^k \tilde{T}^{n-2k}}{m_p^{n-2}} = 0. \quad (\text{A6})$$

Solving eqn. (A6) for \tilde{X}^2 with respect to \tilde{T} gives the deformed Hamilton-Jacobi equation for I

$$\tilde{X}^2 = \alpha \left(\frac{m}{m_p} \right) \tilde{T}^2 - \beta \left(\frac{m}{m_p} \right) m^2 + \gamma \left(\frac{m}{m_p} \right) m \tilde{T} + \sum_{n \geq 3} \frac{C_n \left(\frac{m}{m_p} \right) \tilde{T}^n}{m_p^{n-2}}, \quad (\text{A7})$$

where α, β, γ are dimensionless functions of $\frac{m}{m_p}$ with $\alpha(0) = \beta(0) = 1$ and $\gamma(0) = 0$ which can be determined by the coefficients B , C and C_n^j in eqn. (A6). In flat spacetime with $A_\mu = 0$, the dispersion relation for the scalar field can be found by inserting the positive energy ansatz $\phi = \exp \left(-\frac{ip_\mu x^\mu}{\hbar} \right)$ into eqn. (A2). The resulting equation for p_μ is actually eqn. (A7) with $\tilde{T} = u^\mu p_\mu$ and $\tilde{X}^2 = -p_\mu p^\mu + \tilde{T}^2$, which is exact for flat spacetime with $A_\mu = 0$. Identifying $E = p_\mu u^\mu = \tilde{T}$ and $p^2 = -p_\mu^{(3)} p^{(3),\mu} = -p_\mu p^\mu + \tilde{T}^2 = \tilde{X}^2$, we can produce the MDR for the scalar, eqn. (5), in flat spacetime. On the other hand, the vector field u^μ is chosen to be $\left(\frac{1}{\sqrt{f(r)}}, \vec{0} \right)$ in curved spacetime with the metric (6) and the electromagnetic potential A_μ . In this case, \tilde{T} and \tilde{X}^2 become T and X^2 in eqn. (13). Thus, in the black hole background spacetime, the corresponding deformed Hamilton-Jacobi equation for the scalar field incorporating the MDR, eqn. (5), is given by eqn. (12).

2. Fermionic Field

In the background spacetime with the metric $g_{\mu\nu}$ and the electromagnetic potential A_μ , the effective Lagrangian for a spin-1/2 fermion ψ with the mass m and the charge q incorporating the MDR can be written as

$$\begin{aligned} \mathcal{L}_{eff}^f = & \bar{\psi} \left(i D_\mu^f \gamma^\mu - \frac{m}{\hbar} \right) \psi - \frac{m}{\hbar} \sum_{k \geq 0, j} B_{\mu_1 \dots \mu_k}^j \left(\frac{m}{m_p} \right) \bar{\psi} \gamma^{\mu_1} \dots \gamma^{\mu_k} \psi \\ & + i \sum_{n \geq k \geq 1, j} \left(\frac{\hbar}{i} \right)^{k-1} \frac{C_{\mu_1 \dots \mu_n}^j \left(\frac{m}{m_p} \right)}{m_p^{k-1}} \bar{\psi} D^{f, \mu_1} \dots D^{f, \mu_k} \gamma^{\mu_{k+1}} \dots \gamma^{\mu_n} \psi, \end{aligned} \quad (\text{A8})$$

where extra tensors $B_{\mu_1 \dots \mu_k}^j$ and $C_{\mu_1 \dots \mu_n}^j$ are dimensionless functions of $\frac{m}{m_p}$ with $B_{\mu_1 \dots \mu_k}^j(0) = C_{\mu_1 \dots \mu_n}^j(0) = 0$, j runs over all independent operators of a given dimension, $D_\mu^f = \partial_\mu + \Omega_\mu + \frac{iq}{\hbar} A_\mu$, $\Omega_\mu \equiv \frac{i}{2} \omega_\mu^{ab} \Sigma_{ab}$, Σ_{ab} is the Lorentz spinor generator, ω_μ^{ab} is the spin connection and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The Greek indices are raised and lowered by the curved metric $g_{\mu\nu}$, while the Latin indices are governed by the flat metric η_{ab} . The deformed Dirac equation is

$$\begin{aligned} & \left(i D_\mu^f \gamma^\mu - \frac{m}{\hbar} \right) \psi - \frac{m}{\hbar} \sum_{n \geq 0, j} B_{\mu_1 \dots \mu_n}^j \left(\frac{m}{m_p} \right) \gamma^{\mu_1} \dots \gamma^{\mu_n} \psi \\ & + i \sum_{n \geq k \geq 1, j} \left(\frac{\hbar}{i} \right)^{m-1} \frac{C_{\mu_1 \dots \mu_n}^j \left(\frac{m}{m_p} \right)}{m_p^{k-1}} D^{f, \mu_1} \dots D^{f, \mu_k} \gamma^{\mu_{k+1}} \dots \gamma^{\mu_n} \psi = 0. \end{aligned} \quad (\text{A9})$$

With rotational symmetry, the extra tensors become

$$\begin{aligned} B_{\mu_1 \dots \mu_n}^j &= B_n^j g_{\mu_{i_1} \mu_{i_2}} \dots g_{\mu_{i_{2k-1}} \mu_{i_{2k}}} u_{\mu_{i_{2k+1}}} \dots u_{\mu_{i_n}}, \\ C_{\mu_1 \dots \mu_n}^j &= C_n^j g_{\mu_{i_1} \mu_{i_2}} \dots g_{\mu_{i_{2k-1}} \mu_{i_{2k}}} u_{\mu_{i_{2k+1}}} \dots u_{\mu_{i_n}}, \end{aligned} \quad (\text{A10})$$

where B_n^j and C_n^j are dimensionless functions of $\frac{m}{m_p}$, $j = (k, \mathcal{C})$, $2k \leq n$, and \mathcal{C} denotes any possible permutations of $(1, \dots, n)$, namely (i_1, \dots, i_n) . To obtain the deformed Hamilton-Jacobi equation, the ansatz for ψ is assumed as

$$\psi = \exp \left(\frac{iI}{\hbar} \right) \mathbf{v}, \quad (\text{A11})$$

where \mathbf{v} is a slowly varying spinor amplitude. Substituting eqn. (A11) into eqn. (A9), we find to the lowest order of \hbar

$$\begin{aligned} & X_\mu \gamma^\mu \mathbf{v} + m \mathbf{v} + m \sum_{n \geq 0, k \leq \frac{n}{2}, \mathcal{C}} B_n^j g_{\mu_{i_1} \mu_{i_2}} \dots g_{\mu_{i_{2k-1}} \mu_{i_{2k}}} u_{\mu_{i_{2k+1}}} \dots u_{\mu_{i_n}} \gamma^{\mu_1} \dots \gamma^{\mu_n} \mathbf{v} \\ & + \sum_{n \geq k \geq 1, l \leq \frac{n}{2}, \mathcal{C}} \frac{C_n^j}{m_p^{k-1}} g_{\mu_{i_1} \mu_{i_2}} \dots g_{\mu_{i_{2l-1}} \mu_{i_{2l}}} u_{\mu_{i_{2l+1}}} \dots u_{\mu_{i_n}} X^{\mu_1} \dots X^{\mu_k} \gamma^{\mu_{k+1}} \dots \gamma^{\mu_n} \mathbf{v} = 0, \end{aligned} \quad (\text{A12})$$

where $X_\mu = \partial_\mu I + qA_\mu$. Using $(X_\mu \gamma^\mu)^2 = (X_\mu)^2 = \tilde{T}^2 - \tilde{X}^2$, $(u_\mu \gamma^\mu)(X_\mu \gamma^\mu) = u_\mu X^\mu = -\tilde{T}$, and $(u_\mu \gamma^\mu)^2 = 1$, one could rewrite eqn. (A12) as

$$\left[X_\mu + mC_1 \left(\frac{m}{m_p} \right) u_\mu + h_2 \left(\tilde{T}, \tilde{X}^2 \right) u_\mu + g \left(\tilde{T}, \tilde{X}^2 \right) X_\mu \right] \gamma^\mu \mathbf{v} = - \left[m + mC_2 \left(\frac{m}{m_p} \right) + h_1 \left(\tilde{T}, \tilde{X}^2 \right) \right] \mathbf{v}, \quad (\text{A13})$$

where C_i are dimensionless functions of $\frac{m}{m_p}$ with $C_i(0) = 0$, $h_i \left(\tilde{T}, \tilde{X}^2 \right) = \sum_{2p+q \geq 1} \frac{h_i^{p,q} \left(\frac{m}{m_p} \right) \tilde{T}^q (\tilde{T}^2 - \tilde{X}^2)^p}{m_p^{2p+q-1}}$ and $g_i \left(\tilde{T}, \tilde{X}^2 \right) = \sum_{2p+q \geq 0} \frac{g_i^{p,q} \left(\frac{m}{m_p} \right) \tilde{T}^q (\tilde{T}^2 - \tilde{X}^2)^p}{m_p^{2p+q}}$. The coefficients C_i , $h_i^{p,q}$ and $g_i^{p,q}$ are determined by B_n^j and C_n^j from eqn. (A12). However, the detailed relations between them are irrelevant here. Multiplying both sides of eqn. (A13) from the left by $(X_\mu + mC_1 u_\mu + h_2 u_\mu + gX_\mu) \gamma^\mu$ and then using eqn. (A13) and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ to simplify the RHS, one gets

$$(X^\mu + mC_1 u^\mu + h_2 u^\mu + gX^\mu) (X_\mu + mC_1 u_\mu + h_2 u_\mu + gX_\mu) \mathbf{v} = (m + mC_2 + h_1)^2 \mathbf{v}. \quad (\text{A14})$$

Since \mathbf{v} is nonzero, eqn. (A14) gives

$$\left(\tilde{T}^2 - \tilde{X}^2 - m^2 \right) - m^2 B \left(\frac{m}{m_p} \right) - mC \left(\frac{m}{m_p} \right) \tilde{T} + \sum_{n \geq 2, k \leq \frac{n}{2}, C} \frac{C_n^j \left(\frac{m}{m_p} \right) \left(\tilde{T}^2 - \tilde{X}^2 \right)^k \tilde{T}^{n-2k}}{m_p^{n-2}} = 0, \quad (\text{A15})$$

where

$$\begin{aligned} B &= -C_1^2 + C_2^2 + 2C_2, \\ C &= 2C_1 + 2g^{0,0}C_1 + 2C_1 h_2^{0,1} - 2(1 + C_2) h_1^{0,1}, \\ \sum \frac{C_n^j \left(\tilde{T}^2 - \tilde{X}^2 \right)^k \tilde{T}^{n-2k}}{m_p^{n-2}} &= g \left(\tilde{T}^2 - \tilde{X}^2 \right) + 2(h_2 + gh_2) \tilde{T} - h_1^2 + h_2^2 \\ &+ 2m(g - g^{0,0}) C_1 \tilde{T} + 2mC_1 \left(h_2 - h_2^{0,1} \tilde{T} \right) - 2m(1 + C_2) \left(h_1 - h_1^{0,1} \tilde{T} \right). \end{aligned}$$

It is noted that the form of eqn. (A14) is the same as that of eqn. (A6). Thus, the argument and result below eqn. (A7) can also apply to a spin-1/2 fermion field.

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